# $40^{\text{th}}$ Indian National Mathematical Olympiad - 2026

**Time: 4.5 hours** **January 18, 2026**

**Instructions:**

- **Answer all questions. All questions carry equal marks. Maximum marks: 102.**
- **Each answer should start on a new page, clearly indicating the question number**.
- **No marks will be awarded for stating an answer without justification.**
- **Calculators (in any form), protractors, and electronic devices are not allowed.**
- **Rulers and compasses are allowed. Draw neat and labeled diagrams.**

1. Let $x_1, x_2, x_3, \ldots$ be a sequence of positive integers defined as follows: $x_1 = 1$ and for each $n \geq 1$ we have
$$x_{n+1} = x_n + \lfloor \sqrt{x_n} \rfloor.$$
Determine all positive integers $m$ for which $x_n = m^2$ for some $n \geq 1$. (Here $\lfloor x \rfloor$ denotes the greatest integer less or equal to $x$ for every real number $x$.)

2. Let $f : \mathbb{N} \to \mathbb{N}$ be a function satisfying the following condition: for each $k > 2026$, the number $f(k)$ equals the maximum number of times a number appears in the list $f(1), f(2), \ldots, f(k-1)$. Prove that $f(n) = f(n + f(n))$ for infinitely many $n \in \mathbb{N}$.

   (Here $\mathbb{N}$ denotes the set $\{1, 2, 3, \ldots\}$ of positive integers.)

3. Let $ABC$ be an acute-angled scalene triangle with circumcircle $\Gamma$. Let $M$ be the midpoint of $BC$ and $N$ be the midpoint of the minor arc $\widehat{BC}$ of $\Gamma$. Points $P$ and $Q$ lie on segments $AB$ and $AC$ respectively such that $BP = BN$ and $CQ = CN$. Point $K \neq N$ lies on line $AN$ with $MK = MN$. Prove that $\angle PKQ = 90°$.

4. Two integers $a$ and $b$ are called *companions* if every prime number $p$ either divides both or none of $a, b$. Determine all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that $f(0) = 0$ and the numbers $f(m) + n$ and $f(n) + m$ are *companions* for all $m, n \in \mathbb{N}_0$.

   (Here $\mathbb{N}_0$ denotes the set of all non-negative integers.)

5. Three lines $\ell_1, \ell_2, \ell_3$ form an acute angled triangle $\mathcal{T}$ in the plane. Point $P$ lies in the interior of $\mathcal{T}$. Let $\tau_i$ denote the transformation of the plane such that the image $\tau_i(X)$ of any point $X$ in the plane is the reflection of $X$ in $\ell_i$, for each $i \in \{1, 2, 3\}$. Denote by $P_{ijk}$ the point $\tau_k(\tau_j(\tau_i(P)))$ for each permutation $(i, j, k)$ of $(1, 2, 3)$.

   Prove that $P_{123}, P_{132}, P_{213}, P_{231}, P_{312}, P_{321}$ are concyclic if and only if $P$ coincides with the orthocentre of $\mathcal{T}$.

6. Two decks $\mathcal{A}$ and $\mathcal{B}$ of 40 cards each are placed on a table at noon. Every minute thereafter, we pick the top cards $a \in \mathcal{A}$ and $b \in \mathcal{B}$ and perform a *duel*.

   For any two cards $a \in \mathcal{A}$ and $b \in \mathcal{B}$, each time $a$ and $b$ duel, the outcome remains the same and is independent of all other duels. A duel has three possible outcomes:

   - If a card wins, it is placed back at the top of its deck and the losing card is placed at the bottom of its deck.
   - If $a$ and $b$ are evenly matched, they are both removed from their respective decks.
   - If $a$ and $b$ do not interact with each other, then both are placed at the bottom of their respective decks.

   The process ends when both decks are empty. A process is called a *game* if it ends. Prove that the maximum time a *game* can last equals 356 hours.

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