INMO 2026

Official Solutions

Problem 1. Let $x_1, x_2, x_3, \ldots$ be a sequence of positive integers defined as follows: $x_1 = 1$ and for each $n \geqslant 1$ we have

$$x_{n+1} = x_n + \lfloor \sqrt{x_n} \rfloor.$$

Determine all positive integers m for which $x_n = m^2$ for some $n \geqslant 1$. (Here $\lfloor x \rfloor$ denotes the greatest integer less or equal to x for every real number x.)

Solution. We claim that m satisfies the condition if and only if $m = 2^k$ for some integer $k \geqslant 0$.

The first few terms of $(x_n)_{n\geqslant 1}$ are $\boxed{1}, 2, 3, \boxed{4}, 6, 8, 10, 13, \boxed{16}, 20, 24, 28, 33, 38, 44, 50, 57, \boxed{64}, \ldots$ so we know the claim is true for $m \leqslant 8$. Let $k \geqslant 3$ be a positive integer for which there exists a t such that $x_t = 4^k$. It suffices to show that the first term in the sequence $(x_n)_{n>t}$ which is a perfect square equals 4^{k+1}. Define the sequences $(a_n)_{n\geqslant 1}, (b_n)_{n\geqslant 1}$ as $a_n = \lfloor \sqrt{x_n} \rfloor$ and $b_n = x_n - a_n^2$ for each n. Clearly, $x_n = a_n^2 + b_n$ with $0 \leqslant b_n \leqslant 2a_n$ for each $n \geqslant 1$. We keep track of the pairs (a_n, b_n) for $n > t$ until we reach a square.

Claim. For any positive integer n, we have

$$(a_{n+2}, b_{n+2}) = \begin{cases} (a_n, 2a_n) & \text{if } b_n = 0 \\ (a_n + 1, b_n - 1) & \text{if } 0 < b_n \leqslant a_n \\ (a_n + 1, b_n) & \text{if } a_n < b_n \leqslant 2a_n \end{cases}$$

Proof. We consider each of the three cases as below:

- Suppose $b_n = 0$, then $x_n = a_n^2$ so $x_{n+1} = a_n^2 + a_n$ and $x_{n+2} = a_n^2 + 2a_n < (a_n + 1)^2$ hence $b_{n+2} = 2a_n$ and $a_{n+2} = a_n$.
- Suppose $0 < b_n \leqslant a_n$, then

  $$x_{n+1} = x_n + a_n = a_n^2 + a_n + b_n \leqslant a_n^2 + 2a_n < (a_n + 1)^2 \implies a_{n+1} = a_n \text{ and } b_{n+1} = a_n + b_n$$

  so

  $$x_{n+2} = x_{n+1} + a_{n+1} = a_n^2 + (2a_n + b_n) = (a_n + 1)^2 + (b_n - 1) \in [(a_n + 1)^2, (a_n + 2)^2 - 1]$$

  hence $a_{n+2} = a_n + 1$ and $b_{n+2} = b_n - 1$.
- Finally, suppose $b_n > a_n$, then $x_{n+1} = x_n + a_n = a_n^2 + a_n + b_n = (a_n + 1)^2 + (b_n - a_n - 1)$ so $b_{n+1} = b_n - a_n - 1$ and $a_{n+1} = a_n + 1$, hence

  $$x_{n+2} = x_{n+1} + a_{n+1} = x_n + 2a_n + 1 = (a_n + 1)^2 + b_n < (a_n + 2)^2$$

  so $b_{n+2} = b_n$ and $a_{n+2} = a_n + 1$. This proves the claim. $\square$

Now $x_t = 4^k$, so $x_{t+1} = 4^k + 2^k$ and $x_{t+2} = 4^k + 2 \cdot 2^k$ so $(2^k)^2 = x_t < x_{t+1} < x_{t+2} < (2^k + 1)^2$.

We claim that for each $1 \leqslant j \leqslant 2^k$ we have $b_{t+2j-1} = 2^k - (j - 1)$ and $b_{t+2j} = 2^{k+1}$ and $a_{t+2j-1} = a_{t+2j} = 2^k + j - 1$. We will proceed by induction on $j \geqslant 1$. The base case $j = 1$ is true from the above reasoning. Suppose that the claim holds for some $j < 2^k$, then we know by combining the induction hypothesis and the previous claim that

$$0 < 2^k - (j - 1) = b_{t+2j-1} \leqslant 2^k = a_t \leqslant a_{t+2j-1} \implies b_{t+2j+1} = b_{t+2j-1} - 1 = 2^k - j$$

$$a_{t+2j} = a_t + j - 1 = 2^k + (j - 1) < 2^{k+1} = b_{t+2j} < 2a_{t+2j} \implies b_{t+2j+2} = b_{t+2j} = 2^{k+1}$$

$$0 \notin \{b_{t+2j-1}, b_{t+2j}\} \implies a_{t+2j+1} = a_{t+2j-1} + 1 = 2^k + j \quad \text{and} \quad a_{t+2j+2} = a_{t+2j} + 1 = 2^k + j$$

completing the induction step.

Since $b_\ell \neq 0$ for all $t < \ell < t + 2^k + 1$ we know that none of $x_{t+1}, \ldots, x_{t+2^k}$ are perfect squares. Now $b_{t+2^{k+1}-1} = 2^k - (2^k - 1) = 1$ hence $b_{t+2^{k+1}+1} = b_{t+2^{k+1}-1} - 1 = 0$ so $x_{t+2^{k+1}+1}$ is a perfect square. Also, $a_{t+2^{k+1}-1} = 2^k + 2^k - 1 = 2^{k+1} - 1$ and $b_{t+2^{k+1}-1} \neq 0$ hence $a_{t+2^{k+1}+1} = a_{t+2^{k+1}-1} + 1 = 2^{k+1}$.

Thus, $x_{t+2^{k+1}+1} = 4^{k+1}$ is the next perfect square in $(x_n)_{n>t}$ as desired. $\square$

Problem 2. Let $f : \mathbb{N} \to \mathbb{N}$ be a function satisfying the following condition: for each $k > 2026$, the number $f(k)$ equals the maximum number of times a number appears in the list $f(1), f(2), \dots, f(k-1)$. Prove that $f(n) = f(n + f(n))$ for infinitely many $n \in \mathbb{N}$.

(Here $\mathbb{N}$ denotes the set $\{1, 2, 3, \dots\}$ of positive integers.)

Solution. Let $\mathcal{S} = \{f(1), f(2), f(3) \dots\}$ be the set of all possible values taken by f. We make the following claims:

Claim 1. For each $k > 2026$, we have $f(k+1) = f(k)$ or $f(k+1) = f(k) + 1$.

Proof. Indeed, the maximum frequency of an element in $f(1), f(2), \dots, f(k)$ is at least as much as the maximum frequency of an element in $f(1), f(2), \dots, f(k-1)$ and the increment is 1 if and only if $f(k)$ equals an element with this maximum frequency. $\square$

Claim 2. The set $\mathcal{S}$ is infinite.

Proof. Assume to the contrary. By Claim 1, f is non-decreasing, so the assumption implies f will be eventually constant. Thus, there exist positive integers N, c such that $f(n) = c$ for all $n > N$. Now $f(N+1) = f(N+2) = \dots = f(N+c+1) = c$ so the frequency of c in $f(1), f(2), \dots, f(N+c+1)$ is at least $c+1$, hence $f(N+c+2) \geqslant c+1$, a contradiction! $\square$

Let $\mathcal{T} = \mathcal{S} \setminus \{f(1), f(2), \dots, f(2027)\}$ and for each $n \in \mathcal{T}$, let x_n be the smallest positive integer with $f(x_n) = n$.

Claim 3. We have $f(x_n) = f(x_n + f(x_n)) = n$ for each $n \in \mathcal{T}$.

Proof. Let $n \in \mathcal{T}$ and t be the element with the maximum frequency in the list $f(1), f(2), \dots, f(x_n - 1)$.

Note that t occurs precisely n times in this list of numbers, since $f(x_n) = n$. Note that n does not occur in the list $f(1), f(2), \dots, f(x_n - 1)$ by the minimality of x_n. So $t \neq n$.

We will prove by induction that $0 \leqslant i \leqslant n \implies f(x_n + i) = n$. This is true for $i = 0$, so we show that $f(x_n + i) = n \implies f(x_n + i + 1) = n$ for all $i < n$, which suffices.

Indeed, the element t has frequency n in the list $f(1), f(2), \dots, f(x_n), \dots, f(x_n + i)$ and the element n, which occurs for the first time at $f(x_n)$ and equals $f(x_n + j)$ for each $j \leqslant i$, occurs only $i + 1$ times. Thus, the maximum frequency of an element in the list $f(1), f(2), \dots, f(x_n), \dots, f(x_n + i)$ is n, hence $f(x_n + i + 1) = n$ completing the induction. $\square$

Since $\mathcal{T}$ is infinite by Claim 2, by Claim 3 we have infinitely many positive integers n with $f(n) = f(n + f(n))$, namely, $n = x_m$ for $m \in \mathcal{T}$. $\square$

Remark. Eventually, the sequence becomes

$$\dots \underbrace{t, t, \dots, t}_{t+1 \text{ times}} \ \underbrace{t+1, t+1, \dots, t+1}_{t+2 \text{ times}} \ \underbrace{t+2, t+2, \dots, t+2}_{t+3 \text{ times}} \dots$$

Problem 3. Let ABC be an acute-angled scalene triangle with circumcircle Γ. Let M be the midpoint of BC and N be the midpoint of the minor arc $\widehat{BC}$ of Γ. Points P and Q lie on segments AB and AC respectively such that $BP = BN$ and $CQ = CN$. Point $K \neq N$ lies on line AN with $MK = MN$. Prove that $\angle PKQ = 90^\circ$.

Solution. Reflect N in M to obtain L. Note that $MK = MN = ML \implies \angle LKN = 90^\circ$. Since $BP = BN = BL$ and $CQ = CN = CL$ and

$$\angle LBA + \angle LCA = \angle BLC - \angle BAC = \angle BNC - \angle BAC = (180^\circ - 2\angle BAC),$$

we have

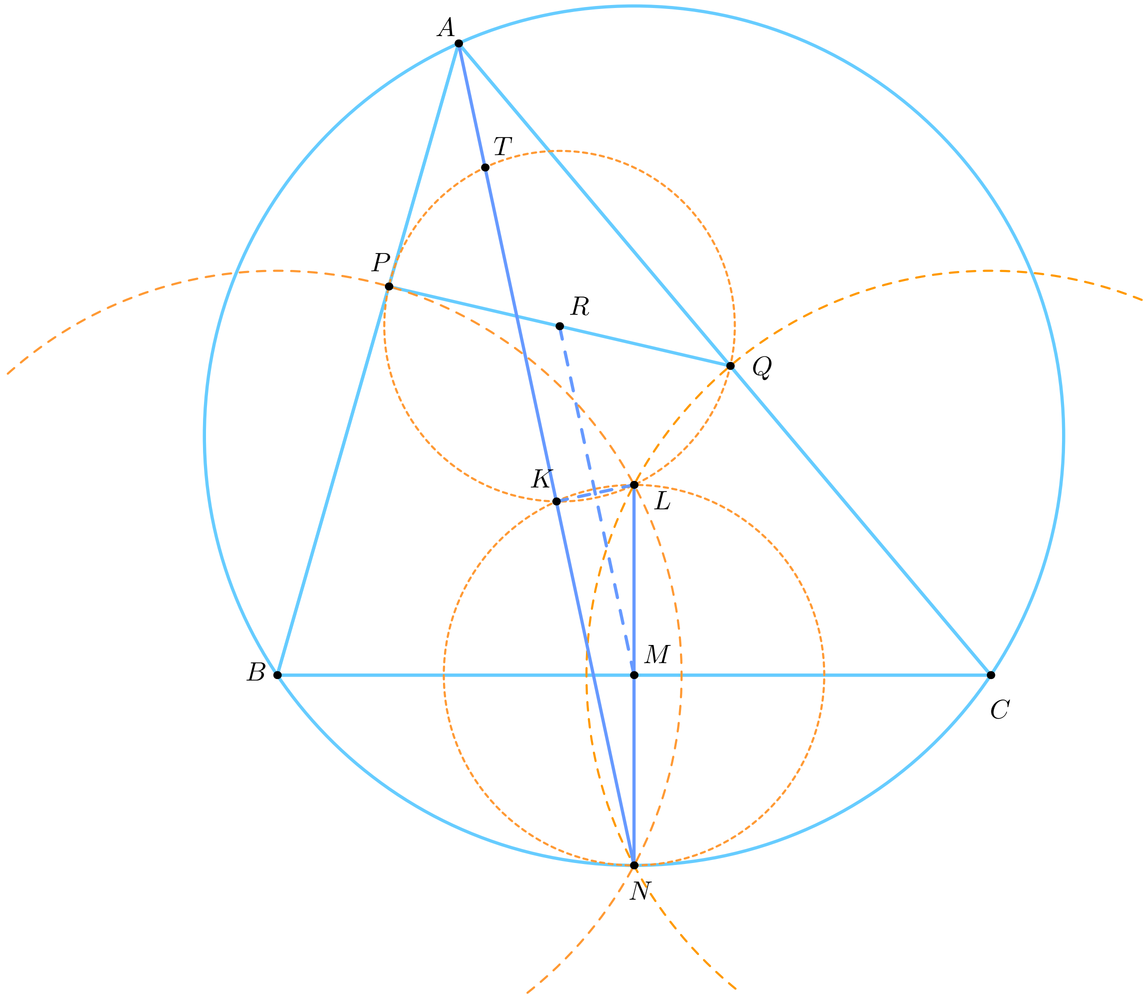
$$\begin{aligned}\angle BLP + \angle CLQ &= \left(90^\circ - \frac{1}{2}\angle LBA\right) + \left(90^\circ - \frac{1}{2}\angle LCA\right)\\ &= 180^\circ - \frac{1}{2}(\angle LBA + \angle LCA)\\ &= 180^\circ - \frac{1}{2}(180^\circ - 2\angle BAC)\\ &= 90^\circ + \angle BAC.\end{aligned}$$

Thus

$$\angle PLQ = 360^\circ - \angle BLC - (\angle BLP + \angle CLQ) = 360^\circ - (180^\circ - \angle BAC) - (90^\circ + \angle BAC) = 90^\circ.$$

Now $BN = CN \implies BP = CQ$ and so $\overrightarrow{BP}, \overrightarrow{CQ}$ are vectors with equal magnitude. Hence the vector $\overrightarrow{BP} + \overrightarrow{CQ}$ is parallel to the internal bisector of angle $\angle BAC$. Let R be the midpoint of PQ, then $\overrightarrow{MR} = \frac{1}{2}\left(\overrightarrow{BP} + \overrightarrow{CQ}\right)$ so $MR \parallel AN$. Thus, the reflection T of L in R lies on line AN as $MN = ML$.

Now $\angle PLQ = 90^\circ$ so LT is the diameter of the circumcircle of triangle PLQ. Since $\angle LKN = 90^\circ$ and T lies on KN, we conclude that $\angle LKT = 90^\circ$, hence K lies on the circle with diameter TL. Thus K lies on the circumcircle of triangle PLQ so $\angle PLQ = 90^\circ \implies \angle PKQ = 90^\circ$ as desired. $\square$



Problem 4. Two integers a and b are called *companions* if every prime number p either divides both or none of a, b. Determine all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that $f(0) = 0$ and the numbers $f(m) + n$ and $f(n) + m$ are *companions* for all $m, n \in \mathbb{N}_0$.

(Here $\mathbb{N}_0$ denotes the set of all non-negative integers.)

Solution 1. We prove the following claim.

Claim. Let p be a prime number that divides $f(x) - f(y)$, then p divides $x - y$.

Proof. Let $n > 0$ be a sufficiently large integer such that $z = np - f(x) > 0$. Then $p \mid f(x) + z \implies p \mid x + f(z)$ and likewise

$$f(y) + z = -(f(x) - f(y)) + (f(x) + z) \equiv 0 \pmod p$$

hence $p \mid y + f(z)$. Together we get

$$x - y = (x + f(z)) - (y + f(z)) \equiv 0 \pmod p$$

as desired. □

Thus every prime factor p of $f(x + 1) - f(x)$ divides 1, so it must be the case that for each $x \geqslant 0$ we have $f(x + 1) - f(x) \in \{-1, 1\}$.

Note that $f(x) = f(x + 2) \implies 3 \mid f(x + 2) - f(x) \implies 3 \mid (x + 2) - x$ which is clearly impossible. Thus, $f(x + 1) - f(x) = f(x + 2) - f(x + 1)$ for all $x \geqslant 0$ hence $f(x + 1) - f(x)$ is a constant for all $x \geqslant 0$. Since f only takes non-negative values, this constant must be 1 hence $f(x + 1) = f(x) + 1$ for all $x \geqslant 0$ so $f(x) = x$ for all $x \in \mathbb{N}$. □

Solution 2. Notice that $f(x) = x$ for all $x \in \mathbb{N}_0$ is a solution as both $f(m)+n$ and $f(n)+m$ equal to $m + n$ for each $m, n \geqslant 0$. We claim that this is the only function which satisfies the above condition. Let us denote the statement as

$$P(x, y) : f(x) + y \text{ and } x + f(y) \text{ are companions.}$$

Plug $y = 0$ to conclude that $f(x)$ and x have the same set of prime divisors for each $x \geqslant 1$. In particular, $f(2^k)$ is a power of 2 (other than 1) for all $k \geqslant 1$ and $f(1) = 1$. Further, let $g(k)$ denote the positive integer for which $f(2^k) = 2^{g(k)}$. Plug $x = 2^{g(k)}$ and $y = 2^k$ to conclude

$$2^{g(g(k))} + 2^k \quad \text{and} \quad 2^{g(k)} + 2^{g(k)} \text{ are companions}$$

hence $2^{g(g(k))} + 2^k$ is a power of 2. Now $2^{|g(g(k))-k|} + 1$ divides $2^{g(g(k))} + 2^k$, so if $g(g(k)) \neq k$, then $2^{|g(g(k))-k|} + 1$ is odd and greater than 1, which is absurd.

So we must have $g(g(k)) = k$ for each $k > 0$. Thus, g is a bijection on positive integers, as clearly g is surjective and for any $a, b \in \mathbb{N}$, we have $g(a) = g(b) \implies a = g(g(a)) = g(g(b)) = b$, hence it is injective as well. Now plug $x = 2^{g(a)}$ and $y = 2^b$ to get $2^a + 2^b$ and $2^{g(a)} + 2^{g(b)}$ are companions. Suppose $b = a + 2$, we conclude that $2^{|g(a+2)-g(a)|} + 1$ must be a power of 5.

Claim 1. The equation $2^x + 1 = 5^y$ has $(2, 1)$ as the only positive integer solution.

Proof. Indeed, taking the equation modulo 5 gives $x \equiv 2 \pmod 4$ as $2^x \equiv 4 \mod 5$. Taking it modulo 3, we get $(-1)^y \equiv (-1)^x + 1 \equiv 2 \pmod 3$ hence y is odd. Now

$$2^x = 5^y - 1 = (5 - 1) \cdot (5^{y-1} + 5^{y-2} + \cdots + 5^1 + 1)$$

but $5^{y-1} + \cdots + 5^1 + 1$ is odd since y is odd, hence $y = 1$ and $x = 2$ is the only possibility. □

By Claim 1, we know that $g(a + 2) \in \{g(a) - 2, g(a) + 2\}$ for each $a \geqslant 1$. However, g is also bijective, so $g(a + 2) = g(a) - 2 \implies g(a + 4) = g(a) - 4$ else $g(a + 4) = g(a + 2) + 2 = g(a)$, which would yield a contradiction.

Similarly, we can conclude that $g(a + 2m) = g(a) - 2m$ for each $m \geqslant 1$, however, g only takes positive values so this will fail to hold for $m > g(a)$. This forces $g(a + 2) = g(a) + 2$ for each $a \in \mathbb{N}$. Together with the fact that g is a bijection, we conclude that $\{g(1), g(2)\} = \{1, 2\}$ and $g(n + 2) = g(n) + 2$ for each $n \geqslant 1$.

Finally, $f(2) + 1$ and $2 + f(1) = 3$ are companions, so $f(2) = 2^2 = 4$ is impossible, hence $f(2) = 2$ and $f(4) = 4$, so $g(n) = n$ for all $n \geqslant 1$. In particular, $f(2^k) = 2^k$ for each $k \geqslant 1$.

We will prove the two claims below before finishing the proof.

Claim 2. Let p be an odd prime and m be a positive integer with $p \mid 2^m - 1$. For each $k \geqslant 0$ we have

$$p^k \mid \frac{2^{p^k m} - 1}{2^m - 1}.$$

Proof. The case $k = 0$ is obvious, so we proceed by induction on $k \geqslant 0$. Now

$$\frac{2^{p^{k+1}m} - 1}{2^m - 1} = \frac{2^{p^{k+1}m} - 1}{2^{p^k m} - 1} \cdot \frac{2^{p^k m} - 1}{2^m - 1}$$

and

$$\frac{2^{p^{k+1}m} - 1}{2^{p^k m} - 1} = \left((2^{p^k m})^{p-1} + (2^{p^k m})^{p-2} + \cdots + (2^{p^k m}) + 1\right) \equiv \underbrace{1 + 1 + \cdots + 1}_{p\text{ times}} \equiv 0 \pmod{p}$$

implies that if the statement is true for any value of k, it is true for $k + 1$ as well. This completes the induction step. □

For each prime p and natural number n, let $v_p(n)$ denote the largest integer j such that $p^j \mid n$.

Claim 3. Fix $y \geqslant 1$. Infinitely many prime numbers p exist such that for some $n > 0$, we have $p \mid 2^n + y$.

Proof. Assume without loss of generality that y is odd, by focusing only on $n > v_2(y)$. Assume to the contrary that only finitely many such primes p exist and enumerate them as $2 < p_1 < p_2 < \cdots < p_k$. Let $\ell \geqslant 1$ be a positive integer that we will choose later and let $M = (p_1 \cdot \cdots \cdot p_k)^\ell$ and $N = M \cdot \phi(M)$ where ϕ denotes the Euler-totient function. By Euler's Theorem, we have $p_i \mid M \mid 2^{\phi(M)} - 1 \mid 2^N - 1$ for each $1 \leqslant i \leqslant k$. Choose $\ell > 1 + \max(v_{p_i}(y+1) \mid 1 \leqslant i \leqslant k)$. Notice that

$$2^N + y = (2^N - 1) + (y + 1)$$

and $v_{p_i}(2^N - 1) > v_{p_i}(y+1)$ for each $1 \leqslant i \leqslant k$ by Claim 2, hence $v_{p_i}(2^N + y) = v_{p_i}(y+1)$ for each i. However, $2^N + y > y + 1$, so it must have a prime factor outside of $p_1, p_2, \ldots, p_k$, yielding the desired contradiction! □

Plug $x = 2^n$ to see that $2^n + y$ and $2^n + f(y)$ are companions for each $n \geqslant 1$ and $y \geqslant 1$. Choose n such that for some prime $p > |f(y) - y|$, we have $p \mid 2^n + y$. This is possible due to Claim 3. Now $p \mid 2^n + y$ and so $p \mid 2^n + f(y) \implies p \mid f(y) - y$. This is only possible when $f(y) = y$. Together with $f(0) = 0$, we conclude that $f(x) = x$ for all $x \in \mathbb{N}_0$.

□

Remark. The following theorems are popular in Olympiad number theory and can be used to simplify each of the above claims.

1. Claim 1 can be resolved by **Zsigmondy's Theorem** which states: For all positive integers a, m, n with $m < n$, the number $a^n + 1$ has a prime factor that does not divide $a^m + 1$ with the exception of $a = 2, m = 1, n = 3$.

2. Claim 2 can be resolved by **Exponent Lifting Lemma (LTE)** which states: For any odd prime p and positive integers a, b, n with $p \mid a - b$, we have

$$v_p(a^n - b^n) = v_p(n) + v_p(a - b).$$

3. Claim 3 can be resolved by **Kobayashi's Theorem** which states: Let M be an infinite set of positive integers and c be a positive integer. If the set of prime numbers p such that $p \mid m$ for some $m \in M$ is finite, then the set of primes q such that $q \mid m + c$ for some $m \in M$ is infinite.

Problem 5. Three lines ℓ_1, ℓ_2, ℓ_3 form an acute angled triangle $\mathcal{T}$ in the plane. Point P lies in the interior of $\mathcal{T}$. Let τ_i denote the transformation of the plane such that the image $\tau_i(X)$ of any point X in the plane is the reflection of X in ℓ_i, for each $i \in \{1,2,3\}$. Denote by P_{ijk} the point $\tau_k(\tau_j(\tau_i(P)))$ for each permutation (i,j,k) of $(1,2,3)$.

Prove that $P_{123}, P_{132}, P_{213}, P_{231}, P_{312}, P_{321}$ are concyclic if and only if P coincides with the orthocentre of $\mathcal{T}$.

Solution. Suppose lines ℓ_2 and ℓ_3 intersect at point A_1, define the points A_2 and A_3 analogously. Without loss of generality, suppose A_1, A_2, A_3 occur in counter-clockwise order on the circumcircle of $\mathcal{T}$. For all points X in the plane of $\mathcal{T}$, let $X_{ij} = \tau_j(\tau_i(X))$ and $X_k = \tau_k(X)$ for $1 \leqslant i,j,k \leqslant 3$. Let $B_i = \tau_i(A_i)$ for each $i \in \{1,2,3\}$.

We prove the following claims for all points X in the plane with $X \notin \{A_1, B_1, A_2, B_2, A_3, B_3\}$. The variants of each claim are valid when the indices are cyclically permuted. Note that lines A_iX_i and B_iX_i are well-defined for each $i \in \{1,2,3\}$ due to X not coinciding with these six points.

Claim 1. $A_1X_{23} = A_1X_{32} = A_1X$ and $\angle X_{32}A_1X = \angle XA_1X_{23} = 2\angle A_2A_1A_3$.

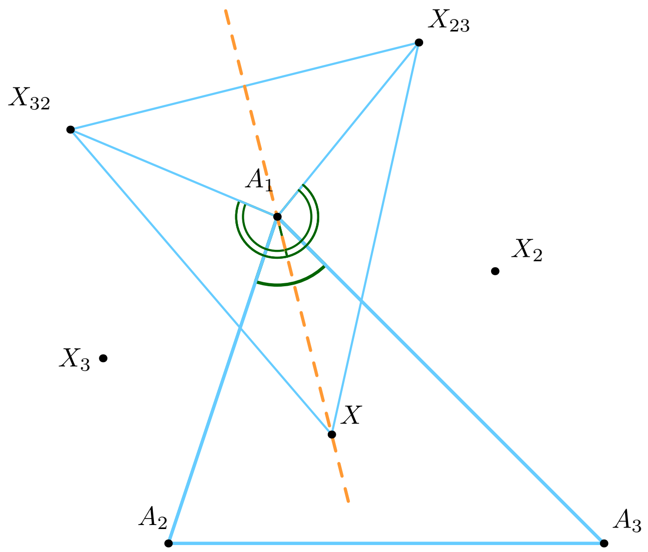
Proof. Denote by $\angle ABC$ the counter-clockwise oriented angle between rays $\overrightarrow{BA}, \overrightarrow{BC}$ modulo $360°$.

Note that $A_1X = A_1X_2 = A_1X_3$ so $A_1X_{32} = A_1X_2 = A_1X$ and $A_1X_{23} = A_1X_3 = A_1X$.

Angle chasing yields

$$\begin{aligned}
\angle X_{32}A_1X &= \angle X_{32}A_1A_2 + \angle A_2A_1X \\
&= \angle A_2A_1X_2 + \angle A_2A_1X && \text{(since } X_2, X_{32} \text{ are reflections in } A_1A_2) \\
&= \angle A_2A_1A_3 + \angle A_3A_1X_2 + \angle A_2A_1X \\
&= \angle A_2A_1A_3 + \angle XA_1A_3 + \angle A_2A_1X && \text{(since } X_2, X \text{ are reflections in } A_1A_3) \\
&= 2\angle A_2A_1A_3
\end{aligned}$$

as desired. Similarly, $\angle XA_1X_{23} = 2\angle A_2A_1A_3$ and the claim follows. □

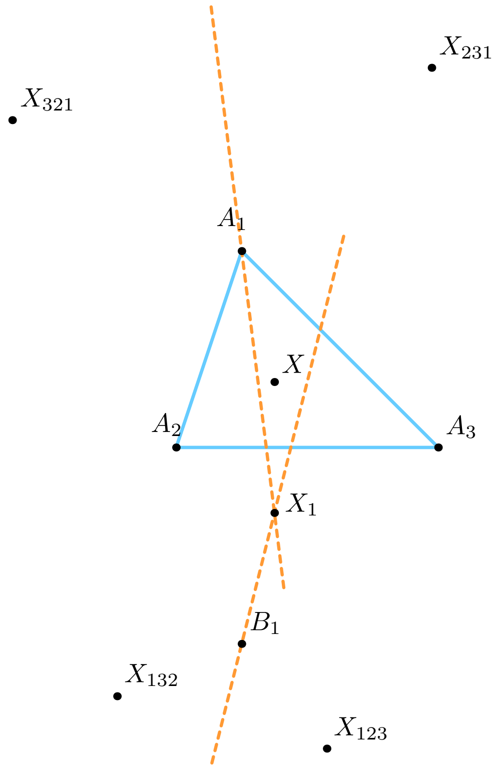


Claim 2. Line A_1X_1 is the perpendicular bisector of $X_{231}X_{321}$.

Proof. By Claim 1 applied to point X_1, we conclude $A_1X_{231} = A_1X_1 = A_1X_{321}$ and $\triangle X_{231}A_1X_1 \cong \triangle X_1A_1X_{321}$ since both are A_1-isosceles triangles, with apex angle at vertex A_1 equal to $2\angle A_2A_1A_3$. Thus X_1 is equidistant from X_{231}, X_{321} and hence A_1, X_1 both lie on the perpendicular bisector of $X_{231}X_{321}$, proving the claim. □

Claim 3. Line B_1X_1 is the perpendicular bisector $X_{132}X_{123}$.

Proof. By Claim 1, line A_1X is the perpendicular bisector of $X_{32}X_{23}$. Reflecting in ℓ_1, we conclude that line B_1X_1 is the perpendicular bisector of $X_{132}X_{123}$, as desired. $\square$



($\Longrightarrow$) Suppose all six points $P_{123}, P_{132}, P_{213}, P_{231}, P_{312}, P_{321}$ lie on a circle denoted Γ with centre denoted O that for some point P in the interior of $\mathcal{T}$. Since P is in the interior of $\mathcal{T}$, we know that $P \notin \{A_1, B_1, A_2, B_2, A_3, B_3\}$, so each of the above claims applies to P.

By Claims 2 and 3, we conclude that the lines A_1P_1 and B_1P_1 are perpendicular bisectors of the chords $P_{231}P_{321}$ and $P_{132}P_{123}$ of Γ respectively. Lines A_1P_1 and B_1P_1 must coincide or P_1 is the center of Γ. Now A_1P_1 and B_1P_1 coincide if and only if $A_1P \perp \ell_1$.

By cyclically permuting the indices, we conclude that

$$P_i = O \text{ or } A_iP_i \perp \ell_i \text{ for each } i \in \{1, 2, 3\}.$$

Now P_1, P_2, P_3 are pairwise distinct, so for at least two indices i, we must have $A_iP \perp \ell_i$, and so P is the intersection of two of the altitudes of $\mathcal{T}$, hence P coincides with the orthocentre of $\mathcal{T}$. $\square$

($\Longleftarrow$) Suppose P is the orthocentre of $\mathcal{T}$. Since $\mathcal{T}$ is acute-angled, P is in the interior of $\mathcal{T}$ hence all claims apply to P. Thus, P, A_i, B_i, P_i are collinear for each $i \in \{1, 2, 3\}$. By Claims 2 and 3 and all their cyclic variants, we conclude that P is equidistant from P_{ijk} and P_{jik}, and that P is equidistant from P_{ijk} and P_{ikj}, for each permutation ijk of $1, 2, 3$.

Since all permutations of $1, 2, 3$ can be achieved by performing transpositions $ijk \mapsto ikj$ and $ijk \mapsto jik$, we conclude that P is equidistant from each of $P_{123}, P_{132}, P_{213}, P_{231}, P_{312}, P_{321}$, proving that the six points all lie on a circle with center P. $\square$

Problem 6. Two decks $\mathcal{A}$ and $\mathcal{B}$ of 40 cards each are placed on a table at noon. Every minute thereafter, we pick the top cards $a \in \mathcal{A}$ and $b \in \mathcal{B}$ and perform a *duel*.

For any two cards $a \in \mathcal{A}$ and $b \in \mathcal{B}$, each time a and b duel, the outcome remains the same and is independent of all other duels. A duel has three possible outcomes:

- If a card wins, it is placed back at the top of its deck and the losing card is placed at the bottom of its deck.
- If a and b are evenly matched, they are both removed from their respective decks.
- If a and b do not interact with each other, then both are placed at the bottom of their respective decks.

The process ends when both decks are empty. A process is called a *game* if it ends. Prove that the maximum time a *game* can last equals 356 hours.

Solution. Let $M(n)$ denote the maximum number of moves a game can last with the initial decks $\mathcal{A}, \mathcal{B}$ consisting of n cards each. We shall prove by strong induction on $n \geqslant 1$ that $M(n) = \frac{n(n^2+2)}{3}$. The base case $n = 1$ is clear, so assume that $M(k) = \frac{k(k^2+2)}{3}$ for $k = 1, 2, \ldots, n-1$. We shall prove the claim for $M(n)$ now.

Estimation. We will prove the bound $M(n) \leqslant M(n-1) + (n^2 - n + 1)$. Consider a game as below.

Let $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ and $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$ be all the cards labelled from top to bottom in each deck. Call a pair (a_i, b_j) *good* if a_i and b_j are evenly matched, and otherwise call it *bad*. We call a *bad* pair (a_i, b_j) *nasty* if a_i and b_j do not interact with each other.

Given that the game ends with both decks empty, it is possible to assign to each a_i a unique b_j for which (a_i, b_j) is a good pair, since there was a moment when the card a_i was removed from its deck along with a card in $\mathcal{B}$.

Thus, there are at least n good pairs among a total of n^2 pairs $(a, b) \in \mathcal{A} \times \mathcal{B}$. Thus, at most $n^2 - n$ bad pairs exist in $\mathcal{A} \times \mathcal{B}$. Enumerate the pair of cards $(a, b) \in \mathcal{A} \times \mathcal{B}$ at the top of their decks at every minute by the sequence $P_1, P_2, P_3, \ldots$ and let r denote the smallest positive integer such that P_{r+1} is a good pair. Each of the pairs $P_1, \ldots, P_r$ is bad.

If $r \geqslant n^2 - n + 1$, then by the Pigeonhole Principle, for some $1 \leqslant i < j < r$, we have $P_i = P_j$ as there are at most $n^2 - n$ bad pairs. However, notice that for any bad pair (a, b), when the duel occurs between a, b and one or both of them are placed at the bottom of their respective decks, the *relative cyclic order* of the cards in each deck is preserved. This means that if the pair (a, b) ever appears again at the top of both decks, the two decks are placed identically to the previous instant in which this happened. Thus, $P_i = P_j$ implies that the decks $\mathcal{A}, \mathcal{B}$ are arranged exactly the same way at the i-th minute and the j-th minute respectively, so $P_{i+k} = P_{j+k}$ for all $k \geqslant 0$, and the process becomes periodic, hence all configurations after the j-th minute have occured previously in the game. But since no cards have been removed from play by the j-th minute, the sequence of duels never yields a good pair and so no cards are ever removed from play and the game never actually ends, a contradiction!

This proves that $r \leqslant n^2 - n$ hence after $r + 1$ moves, we are down to two decks of $n - 1$ cards each. The game thereafter can last at most $M(n-1)$ moves, so the total number of moves the game lasted equals $r + 1 + M(n-1) \leqslant (n^2 - n + 1) + M(n-1)$. Since this is true for *any* game with decks of size n, we conclude that $M(n) \leqslant n^2 - n + M(n-1)$. Together with the induction hypothesis, we have

$$M(n) \leqslant n^2 - n + 1 + \frac{(n-1)((n-1)^2+2)}{3} = \frac{n(n^2+2)}{3}.$$

□

It remains to show that it is possible for the game to last exactly $\frac{n(n^2+2)}{3}$ moves.

Construction. Suppose we have decks $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}, \mathcal{B} = \{b_1, b_2, \ldots, b_n\}$ with the following relations:

- (a_i, b_j) is a good pair if and only if $i + j = n + 1$.
- (a_i, b_j) is a nasty pair if and only if $i + j = n$.

- (a_i, b_j) is a bad (but not nasty) pair if and only if $i + j \notin \{n, n+1\}$ and for all such pairs, a_i is the winner of the duel.

Let m_n denote the number of moves the process described above lasts (set $m_n = \infty$ if the process is not a game). We shall prove that $m_n = \frac{n(n^2+2)}{3}$. The base case $n = 1$ is clear, so we proceed by induction on $n \geqslant 1$ assuming the statement holds for each $1 \leqslant k < n$.

Notice that for any $i < n$, after $i(n-1)$ moves, the decks are (from top to bottom) $\mathcal{A} = \{a_{i+1}, \ldots, a_n, a_1, \ldots, a_i\}, \mathcal{B} = \{b_{n-i+1}, \ldots b_n, b_1, \ldots, b_{n-i}\}$. Thus, after $(n-1)^2$ moves have happened, the decks are $\mathcal{A} = \{a_n, a_1, \ldots, a_{n-1}\}$ and $\mathcal{B} = \{b_2, b_3, \ldots, b_n, b_1\}$ and finally after another $n-1$ moves we reach $\mathcal{A} = \{a_n, a_1, \ldots, a_{n-1}\}$ and $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$. Now (a_n, b_1) is a good pair hence removed from play.

Thus after $(n-1)^2 + (n-1) + 1 = n^2 - n + 1$ moves, we reach the state with decks of size $n-1$ labelled $\mathcal{A} = \{a_1, \ldots, a_{n-1}\}$ and $\mathcal{B} = \{b_2, b_3, \ldots, b_n\}$. If we now relabel each b_k with b_{k-1}, the relations between the (a, b) pairs described above correspond to the same relations described in case of decks of size $n-1$. The induction hypothesis yields $m_n = n^2 - n + 1 + m_{n-1}$ as desired. □

Hence, when $n = 40$, the maximum time a game lasts equals $\frac{1}{60} \cdot \frac{40 \times 1602}{3} = 356$ hours. □

———00000———