

M1

Time: 3 Hours

Questions: 30

Max Marks: 100

INSTRUCTIONS

1. Use of mobile phones, smartphones, tablets, calculators, programmable wrist watches or any other electronic devices is **STRICTLY PROHIBITED**. Only ordinary pens and pencils are allowed inside the examination hall.
2. The correction is done by machines through scanning. On the OMR Sheet, darken bubbles completely with a **black or blue ball pen**. Please **DO NOT use pencil or a gel pen**. Darken the bubbles completely, only after you are sure of your answer; else, erasing may lead to the OMR sheet getting damaged and the machine may not be able to read the answer.
3. The name, email address, and date of birth entered on the OMR sheet will be your login credentials for accessing your score.
4. Incompletely, incorrectly or carelessly filled information may disqualify your candidature.
5. Each question has a one or two digit number as answer. The first diagram below shows improper and proper way of darkening the bubbles with detailed instructions. The second diagram shows how to mark a 2-digit number and a 1-digit number.

INSTRUCTIONS

1. "Think before your ink".
2. Marking should be done with Blue/Black Ball Point Pen only.
3. Darken only one circle for each question as shown in Example Below.

WRONG METHODS


CORRECT METHOD


4. If more than one circle is darkened or if the response is marked in any other way as shown "WRONG" above, it shall be treated as wrong way of marking.
5. Make the marks only in the spaces provided.
6. Carefully tear off the duplicate copy of the OMR without tampering the Original.
7. Please do not make any stray marks on the answer sheet.

Q. 1	Q. 2
47 ☐0☐0 ☐1☐1 ☐2☐2 ☐3☐3 ☒4☐4 ☐5☐5 ☐6☐6 ☐7☒7 ☐8☐8 ☐9☐9	05 ☒0☐0 ☐1☐1 ☐2☐2 ☐3☐3 ☐4☐4 ☒5☐5 ☐6☐6 ☐7☐7 ☐8☐8 ☐9☐9

6. The answer you write on OMR sheet is irrelevant. The darkened bubble will be considered as your final answer.
7. Questions 1 to 10 carry 2 marks each; questions 11 to 20 carry 3 marks each; questions 21 to 30 carry 5 marks each.
8. All questions are compulsory.
9. There are no negative marks.
10. Do all rough work in the space provided below for it. You also have blank pages at the end of the question paper to continue with rough work.
11. After the exam, you may take away the Candidate's copy of the OMR sheet.
12. Preserve your copy of OMR sheet till the end of current olympiad season. You will need it later for verification purposes.
13. You may take away the question paper after the examination.

Note:

1. $\gcd(a, b)$ denotes the greatest common divisor of integers a and b .
2. $\text{lcm}(a, b)$ denotes the least common multiple of integers a and b .
3. Unless otherwise stated all numbers are written in decimal notation.

Questions

1. Let $ABCD$ be a quadrilateral in the xy -plane with AB parallel to CD and $AD = BC$. Suppose $A = (0, 0)$, $B = (10, 0)$, $C = (8, 5)$ and $D = (a, b)$. Determine the value of a^2b .
2. A function is defined on the set of positive integers such that if n is an odd integer, $f(n) = n - 1$ and if n is an even integer, $f(n) = n^2 - 1$. Determine the sum of all possible values of n such that $f(f(n)) = 99$.
3. Find the number of positive integers n less than or equal to 100 such that n is not divisible by any prime number other than 2 or 3.
4. The six faces of a cubical die are numbered with $2^0, 2^1, 2^2, 2^3, 2^4, 2^5$ in such a way that the product of the numbers on any pair of opposite faces is 2^5 . Two such dice are stacked one on top of another. If N is the greatest possible sum of the 9 visible numbers (for all such arrangements of dice), find the sum of the squares of the digits of N .
5. Let N be the coefficient of x^{2025} in the expansion of

$$(x + 1)(x^2 + 3)(x^4 + 5)(x^8 + 7) \cdots (x^{1024} + 21).$$

What is the remainder when N is divided by 100?

Space for rough work

6. The sum of four distinct prime numbers is 240. If none of the four primes is greater than 70, what is the smallest of the four numbers?
7. How many positive integers $n \leq 100$ are divisible by all positive integers i such that $i^3 \leq n$?
8. Consider a 2×3 rectangle made of 6 unit squares. In how many ways can we fill up the six cells using the numbers 1, 2, 3, 4, 5, 6, one in each cell, such that any two numbers in adjacent cells (that is, in cells that share a common side) are coprime to each other?
9. Find the largest integer n such that a square of side length n is contained in a circular disc of area 1000.

Space for rough work

10. Find the largest positive integer n for which the inequality $\sum_{k=1}^{2n} (-1)^k k^2 < 100$ holds.

11. Let m be a positive integer satisfying the equation

$$5(2m+1)(2m+3)(2m+5) = \overline{ababab}$$

where a and b represent different digits and $\overline{ababab}$ is a six digit number. What is the value of $m + a + b$?

12. Find the numbers of ordered pairs (m, n) where m and n are positive integers less than or equal to 20000 such that $m^2 + n^4$ is a power of 2.

Space for rough work

13. In a convex quadrilateral $ABCD$, the lengths of the diagonals are 12 and 16 and the line segments joining the midpoints of the opposite sides are of equal length. What is the maximum possible area of the quadrilateral $ABCD$?
14. The side AB of a square $ABCD$ is 1 and it is also a chord of a circle S . The side CD does not intersect S . The length of the tangent CK , drawn from C to S at the point K is 2. If d is the diameter of S , then calculate d^2 .
15. If a, b, c, d are positive integers such that

$$17(abcd + ab + ad + cd + 1) = 20(bcd + b + d),$$

find $a^2 + b^2 + c^2 + d^2$.

Space for rough work

16. If

$$1 - \frac{1}{2 + \frac{1}{3 + \frac{1}{4 + \frac{1}{5 + \frac{1}{6 + \frac{1}{7}}}}}} = \frac{1}{x_1 + \frac{1}{x_2 + \frac{1}{x_3 + \frac{1}{x_4 + \frac{1}{x_5 + \frac{1}{x_6 + \frac{1}{x_7}}}}}}$$

where $x_1, x_2, \dots, x_7$ are positive integers, find $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$.

17. There are 100 cards in a box which are numbered from 1 to 100. While being blindfolded, Mainak is going to draw one or more cards from the box. After that, he will remove his blindfold and multiply together the numbers on these cards. Mainak wants the product of the numbers on the cards drawn to be a multiple of 6. How many cards does he need to draw to make sure that this will happen?
18. In the plane let the positive end of the x -axis be directed towards East and the positive end of the y -axis be directed towards North. Suppose you are at $(0, 0)$ and you want to go to $(7, 12)$. At every move you are allowed to move unit length towards East or unit length towards North from your current position but you are not allowed to visit any point (h, k) where both h, k are odd. Find the number of such paths n .

Space for rough work

19. Find the number of ordered pairs (m, n) where m and n are positive integers such that $1 \leq m < n \leq 50$ and the product mn is a perfect square.
20. How many four digit numbers $\overline{abcd}$, with non-zero digits a, b, c, d in base 10, are there such that $a + c = bd$ and $b + d = ac$?
21. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying $4f(3 - x) + 3f(x) = x^2$ for any real x . Find the value of $f(27) - f(25)$ to the nearest integer.

Space for rough work

22. Three girls G_1, G_2, G_3 , each read four stories S_1, S_2, S_3, S_4 and discuss which ones they like. No story is liked by all the three. For each of the three pairs of the girls, there is at least one story which is liked by the pair and not liked by the third. Let n be the number of ways in which this is possible. Find the sum of the squares of the digits of n .

23. Let P be a point in the interior of a triangle ABC and let AP, BP, CP meet the sides BC, CA, AB in D, E, F respectively. If

$$\frac{BP}{PE} = \frac{5}{2}, \frac{CP}{PF} = \frac{7}{3}, \text{ and } \frac{AP}{PD} = \frac{p}{q}$$

where p, q are natural numbers and $\gcd(p, q) = 1$, find $p + q$.

24. If a and b are positive integers satisfying $4^a + 4a^2 + 4 = b^2$, what is the maximum possible value of $a + b$.

Space for rough work

25. How many natural numbers $n \leq 105$ are there such that $7|2^n - n^2$?
26. Let ABC be a triangle, D be the midpoint of side BC , O be the circumcenter and H be the orthocenter. If the triangle ODH is equilateral with side length equal to 6 and the area of the triangle ABC can be written as $a\sqrt{b}$, where a, b are positive integers and b is not divisible by the square of any prime, find $a + b$.
27. Consider the collection M of all ordered pairs (a, b) of positive integers a and b which satisfy

$$ab = 406 + 11 \cdot \text{lcm}(a, b) + 7 \cdot \text{gcd}(a, b).$$

What is the smallest possible value of $a + b$?

Space for rough work

28. There are 10 members in a delegation. No two of them have the same height. Let N be the number of ways in which they can stand in a line for a photograph such that

- the leftmost person is the shortest,
- the rightmost person is the tallest, and
- in the line between the shortest and tallest person, there is exactly one person who is shorter than both of his immediate neighbours.

If N can be written as $100a + b$ where a and b are positive integers less than 100, find $a + b$.

29. Let ABC be an isosceles triangle with sides 13, 13 and 10. The tangents to the incircle, drawn parallel to the sides intersect the sides in points D, E, F, G, H, K which form a hexagon. If the area of the hexagon $DEFGHK$ is $m + \frac{n}{l}$, where m, n, l are positive integers with $n < l$ and $\gcd(n, l) = 1$, what is $m + n + l$?

30. The vertices of a regular dodecagon (a polygon with 12 sides) are coloured either blue or red. Let N be the number of all possible colourings such that no three points of the same colour form the vertices of an equilateral triangle, and no four points of the same colour form the vertices of a square. If N can be written as $N = 100p + q$ where p, q are two positive integers less than 100, find $p + q$.

Space for rough work