INMO 2025

Official Solutions

January 2025

Problem 1. Consider the sequence defined by $a_1 = 2, a_2 = 3$, and

$$a_{2k+2} = 2 + a_k + a_{k+1} \quad \text{and} \quad a_{2k+1} = 2 + 2a_k$$

for all integers $k \geqslant 1$. Determine all positive integers n such that $\frac{a_n}{n}$ is an integer.

Answer. $\dfrac{a_n}{n} \in \mathbb{N}$ if and only if $n + 1 = 2^r$ for some positive integer $r \geq 1$.

Solution 1. We first show that if $\frac{a_n}{n} \in \mathbb{N}$ then it must equal 2. Before we do that we rewrite the recurrence as follows for convenience:

$$a_{2k+1} = 2 + 2a_k \tag{1}$$
$$a_{2k} = 2 + a_k + a_{k-1} \tag{2}$$

where (1) holds for $k \geq 1$ and (2) for $k \geq 2$. Consequently,

$$a_{2k+1} - a_{2k} = a_k - a_{k-1} \tag{3}$$
$$a_{2k} - a_{2k-1} = a_k - a_{k-1} \tag{4}$$

holds for all $k \geq 2$. Now, we are ready to prove two observations from the above equations:

(a) **Observation 1:** a_n is even if and only if n is odd.

Proof. Observe that $a_1 = 2$, and for all $k \geq 1$, we have

$$a_{2k+1} = 2a_k + 2$$

and thus a_n is even whenever n is odd. Thus, we just need to prove that a_{2k} is odd for all $k \geq 2$ since $a_2 = 3$ is already odd.

Now, if we have proven that for all $1 \leq i < 2k - 1$, a_i and a_{i+1} have opposite parities. Thus,

$$a_{2k} \equiv a_k + a_{k-1} \equiv 1 \pmod 2$$

Thus, a_{2k+2} is also odd and our inductive claim follows! □

(b) **Observation 2:** $a_k - a_{k-1} \in \{1, 3\}$ for all $k \geq 2$.

Proof. Observe that by repeatedly using (3) and (4), we have that

$$a_k - a_{k-1} \in \{a_3 - a_2, a_2 - a_1\} = \{6 - 3, 3 - 2\} = \{1, 3\}.$$

□

In particular, $a_n \leq 3(n - 1) < 3n$, so $\frac{a_n}{n} < 3$. Thus, if it is an integer, $a_n/n \in \{1, 2\}$. But by (a) above, it follows that $a_n \neq n$, so if $a_n/n \in \mathbb{N}$ then $a_n/n = 2$ as claimed above. Again, by (a), it follows that n is odd.

Definition: We say an integer $n > 1$ is *good*, if $a_{n-1} = 2n - 2$.

Claim 1: If $k > 2$ is an even integer then k is *good* if and only if $\frac{k}{2}$ is *good*.

Proof. From the above part, we know that k must be even for it to be *good*. Now, let $k = 2j$:

$$k \textit{ is good} \iff 2k - 2 = 2 + 2a_{j-1} \iff 2k - 4 = 2a_{j-1} \iff 2j - 2 = a_{j-1} \iff j \textit{ is good}$$

□

Claim 2: $k > 1$ is good if and only if k is a power of 2.

Proof. If $k = 2^r \cdot s$ where $s > 1$ and odd then we have that k is good if and only if s is good but s is odd and thus cannot be good. Similarly, if $k = 2^r$ then it's good if and only if $k = 2$ is good which is indeed true. Thus, k is good if and only if k is a power of 2. □

Thus, we can conclude that

$$n \mid a_n \iff n + 1 \textit{ is good} \iff n + 1 = 2^r \text{ for some integer } r \geq 1$$

This means that $n \mid a_n$ if and only if $n + 1$ is a power of 2. □

Solution 2.

Claim. $n < a_n \leq 2n$ for all n with equality iff n is one less than a power of 2.

Proof. We will prove the claim via induction. Observe that it is clearly true for the bases cases i.e. $n = 1$ and $n = 2$. Now, otherwise, $a_{2k+2} = 2 + a_k + a_{k+1}$ inductively we have

$$2k + 2 < 2 + k + k + 1 < 2 + a_k + a_{k+1} \leq 2 + 2k + 2(k + 1) = 2(2k + 2)$$

Observe since k and $k + 1$ cannot both be one less than powers of 2, the final inequality is indeed strict. Similarly $a_{2k+1} = 2a_k + 2$, thus

$$2k + 1 < 2k + 2 < 2a_k + 2 \leq 2(2k + 1).$$

The final inequality is actually an equality iff $k = 2^t - 1$ for some t, which is equivalent to $2k + 1 = 2^{t+1} - 1$. □

Thus, we get that $\frac{a_n}{n}$ is an integer iff n is one less than a power of 2. □

Problem 2. Let $n \geq 2$ be a positive integer. The integers $1, 2, \cdots, n$ are written on a board. In a move, Alice can pick two integers written on the board $a \neq b$ such that $a + b$ is an even number, erase both a and b from the board and write the number $\frac{a+b}{2}$ on the board instead. Find all n for which Alice can make a sequence of moves so that she ends up with only one number remaining on the board.
Note. When $n = 3$, Alice changes $(1, 2, 3)$ to $(2, 2)$ and can't make any further moves.

Answer. Alice can make a sequence of moves so that she ends up with only one number remaining on the board for all positive integers n other than $2, 3, 4$, and 6.

We say that Alice *wins* for an integer n if there is a sequence of moves that she can make so that she ends up with only one number remaining on the board when she had started with $1, 2, \ldots, n$ on the board. Otherwise, we say Alice *loses* for the integer n.

The following three solutions show that Alice can win when $n \neq 2, 3, 4, 6$. The fact that Alice cannot win in these 4 cases is shown separately at the end. We use multi-sets to denote configurations on the board. We draw $c_1 \mapsto c_2$ if Alice can go from configuration c_1 to configuration c_2 in one move and $c_1 \longmapsto c_2$ if she can reach c_2 in some sequence of moves. Also, let $[m] = \{1, 2, \cdots m\}$ for all naturals m.

Alice can ensure that only one number remains when $\mathbf{n \neq 2, 3, 4, 6}$:

Solution 1. For any $n \geq 2$, we have $[n-1] \cup \{n+1\} \mapsto [n-2] \cup \{n\}$, thus, by repeating the above, for any $n \geq 3$, we have $[n-2] \cup \{n\} \longmapsto \{2\}$. Thus, for any $n \geq 5$, we have

$$[n] = ([n-3] \cup \{n-1\}) \cup \{n-2, n\} \longmapsto \{2, n-2, n\}$$

Now, if $n \neq 6$, then these 3 numbers are not in an AP. Now, if

- **n is odd:** $\{2, n-2, n\} \mapsto \{2, n-1\}$ and both the numbers are even and Alice ends up with exactly one number for all odd $n \geq 5$.
- **n is even:** Atleast one of n and $n+2$ is divisible by 4. Thus, Alice can make a move such that both the numbers on the board are even and can end up with one final number as long as $n - 2 \neq \frac{n+2}{2}$. This only happens when $n = 6$. Thus, Alice can end up with exactly one number for all even $n \geq 8$. □

Solution 2. Before beginning the solution, we have the following two lemmas:

Lemma 1. If Alice can *win* for a number n such that the final number of on the board is $1 \leq m \leq n$, then she can also *win* for the number n such that the final number is $n+1-m$.

Proof. Alice can follow the strategy that she used to reach m but in reverse. More concretely, in the j th move: if Alice had earlier played $(a, b) \mapsto (c)$, she can now make the move $(n+1-a, n+1-b) \mapsto (n+1-c)$.

Since, initially we have $(1, 2, \cdots n) = (n+1-1, n+1-2, \cdots n+1-n)$, she can always play as described above. Thus, the final number on the board would now be $n+1-m$. □

Lemma 2: If Alice can reach the final number m when starting with $(1, \cdots, n)$ then she can reach the final number $m+t$ if she starts with $(t+1, t+2, \cdots t+n)$ for any positive integer t.

Proof. Alice can follow the same strategy that she had earlier but shifted by t. So, in the jth move: if Alice had earlier played $(a, b) \mapsto (c)$, she can now make the move by t as: $(a+t, b+t) \mapsto (c+t)$ and this is always possible. □

Now, we begin the main solution: First observe that for any odd $n > 3$, we have the following process:

$$(1, 2, \ldots, k, k+2, n-1) \mapsto (1, 2, \ldots, k-1, k+1, n-1) \mapsto \cdots \mapsto (2, n-1) \mapsto \left(\frac{n+1}{2}\right)$$

Letting $k = n - 2$ gives us that Alice wins for n.

Now, let $n = 4k$ for some $k > 1$. If we divide $4k$ into two parts of $4k - 1$ and 1, then the first $4k - 1$ numbers leave $2k$ by the previous argument and the second part is just $4k$ i.e.

$$(1, 2, \ldots, 4k - 1, 4k) \longmapsto (2k, 4k) \mapsto (3k)$$

Thus, Alice also wins if the number on the board is $4k$ and can end up with the number $3k$ on the board. Similarly, Alice can also choose to end up with the number $k + 1$ on the board.

Now, let $n = 8k + 2$, for $k > 0$. Then, dividing into the last $8k$ numbers and the first 2 numbers, we get

$$(1, \ldots, 8k + 2) \mapsto (1, 2, 6k + 2) \mapsto (1, 3k + 2)$$

and Alice wins if k is odd. Now, alternately,

$$(1, \ldots, 8k + 2) \mapsto (1, 2, 2k + 3) \mapsto (2, k + 2)$$

and Alice wins if k is even.

Thus, Alice wins for all numbers of the form $8k + 2$. Now, for numbers of the form $8k + 6$, we repeat the previous trick:

$$(1, \ldots, 8k + 6) \mapsto (1, 2, 6k + 5) \mapsto (2, 3k + 3)$$

and Alice wins if k is odd. Similarly, for k even, we get

$$(1, \ldots, 8k + 6) \mapsto (1, 2, 2k + 4) \mapsto (1, k + 3)$$

and Alice wins. Thus, all the cases are covered and Alice wins for all $n > 6$ and $n = 5$. $\square$

Solution 3. Let T be the set of n for which Alice has a winning strategy. Also, for any $t \in T$, let $S_t = \{s \mid [t] \longmapsto \{s\}\}$.

We extend the game to allow $n = 1$ and declare that Alice wins by making no moves. Thus, $1 \in T$ and $S_1 = \{1\}$.

Lemma 1. If $x \in T$ and $y \in S_x$ then $x + 1 - y \in S_x$ as well.

Proof. Alice can follow the strategy that she use to reach y but in reverse i.e. if she used to make a move $(a, b) \mapsto (c)$, she can instead do $(x + 1 - a, x + 1 - b) \mapsto (x + 1 - c)$ and finally she will be left with $x + 1 - y$ as desired. $\square$

Lemma 2. If $x, 2y \in T$, then $x + 2y \in T$.

Proof. Let $a \in S_x$ and $b \in S_{2y}$. Thus, we also have $2y + 1 - b \in S_{2y}$. WLOG $x + a \equiv b \pmod 2$ since otherwise, we could have replaced b with $2y + 1 - b$.

Now, using the first x numbers, we are left with the number a on the board. Now, using the numbers $x + 1, \cdots x + y$, Alice can ensure that she is left with the number $x + b$ on the board. Thus, she can finally perform the move $(a, x + b) \mapsto \frac{x+a+b}{2}$.

$$[x + y] \longmapsto \{a, x + 1, x + 2, \cdots x + y\} \longmapsto \{a, x + b\} \mapsto \left\{\frac{a + b + x}{2}\right\} \qquad \square$$

Lemma 3. $5, 7, 8, 10, 12,$ and 14 are in T with $3 \in S_5$, $4 \in S_7$, $3 \in S_8, S_{10}, S_{14}$ and $7 \in S_{12}$.

Proof. We will show a strategy for each as follows: ($[n]$ stands for the set $\{1, 2, \ldots, n\}$)

5:

$$[5] \mapsto \{2, 2, 4, 5\} \mapsto \{2, 3, 5\} \mapsto \{2, 4\} \mapsto \{3\}$$

7: (The first move is composed of the first 3 moves for $[5]$)

$$[7] \longmapsto \{2, 4, 6, 7\} \mapsto \{2, 5, 7\} \mapsto \{2, 6\} \mapsto \{4\}$$

8: (The first move is composed of the moves for [7])

$$[8] \longmapsto \{4, 8\} \mapsto \{6\}$$

Thus, $6 \in S_8 \implies 3 \in S_8$ by Lemma 1.

10: (The first move is composed of the moves for [8] which lead to 3)

$$[10] \longmapsto \{3, 9, 10\} \mapsto \{6, 10\} \mapsto \{8\}$$

Thus, $8 \in S_{10} \implies 3 \in S_{10}$ by Lemma 1.

12: (In the first move we use the idea from Lemma 2 and that $3 \in S_5, 4 \in S_7$)

$$[12] \longmapsto \{3, 5 + 4\} \mapsto \{6\}$$

Now, since $6 \in S_{12}$, we also have $7 \in S_{12}$ by Lemma 1.

14: (The first move is composed of the moves for 12 which lead to 7)

$$[14] \longmapsto \{7, 13, 14\} \mapsto \{10, 14\} \mapsto \{12\}$$ □

Now, observe that using Lemma 2 and the fact that we have $8, 10, 12, 14 \in T$, we get that $2n \in T$ for all $n \geq 4$ as for any even number > 8, we can keep subtracting 8 from it till we hit one of 8, 10, 12 or 14 and Lemma 2 tells us that we can add even numbers within T freely to one another. Finally using $a = 1$ and $2b$ as any even integer atleast 8, we have $2b + 1$ works for all odd integers atleast 9.

Thus, finally we have that Alice has a winning strategy for all positive integers with the possible exception of $2, 3, 4, 6$. □

Solution 4. We begin with the following lemma:

Lemma 1. If $[n] \mapsto (n - 2)$ then $[n + 3] \mapsto (n + 1)$.

Proof. We have

$$[n + 3] \mapsto \{n - 2, n + 1, n + 2, n + 3\} \mapsto \{n, n + 2\} \mapsto \{n + 1\}.$$ □

Now, note that $[5] \mapsto \{3\}$ and $[10] \mapsto \{8\}$ as in Solution 3, and

$$[9] \mapsto \{1, 4, 7, 8, 9\} \mapsto \{1, 6, 7, 9\} \mapsto \{4, 6, 9\} \mapsto \{5, 9\} \mapsto \{7\},$$

where the first step is to take $\{2, 3, 4, 5, 6\} \mapsto \{4\}$ via the procedure for [5]. Combining these three with the Lemma 1, we have that Alice wins for all $n \geq 8$ and also for $n = 5$. Finally, we see that Alice wins for $n = 7$ via a direct check as in Solution 3.

Solution 5. We have the following lemma:

Lemma 1. If $[n] \mapsto \{n - 4, n - 2\}$ then $[n + 2] \mapsto \{n - 2, n\}$.

Proof. We have

$$[n + 2] \mapsto \{n - 4, n - 2, n + 1, n + 2\} \mapsto \{n - 2, n - 1, n + 1\} \mapsto \{n - 2, n\}.$$ □

Now, we have

$$[7] \mapsto \{1, 2, 3, 4, 6, 6\} \mapsto \{1, 2, 3, 5, 6\} \mapsto \{1, 3, 4, 5\} \mapsto \{2, 4, 5\} \mapsto \{3, 5\},$$

so together with Lemma 1, we get that $[n] \mapsto \{n - 4, n - 2\}$ for all odd $n \geq 7$.

Hence, for all odd $n \geq 7$, $[n] \mapsto \{n - 4, n - 2\} \mapsto \{n - 3\}$ and $[n + 1] \mapsto \{n - 3, n + 1\} \mapsto \{n - 1\}$. Thus, we establish that Alice wins for all $n \geq 7$. We verify $n = 5$ via a direct check as before.

Alice cannot win when $\mathbf{n = 2, 3, 4, 6}$**:**
We will make some without loss of generality remarks for the cases of $n = 4, 6$. These

would follow since if Alice has a winning strategy with moves $(a, b) \mapsto (c)$ then she can also do it in reverse with moves of the form $(n + 1 - a, n + 1 - b) \mapsto (n + 1 - c)$. Thus, she could play the moves symmetrically as well.

Now, we show that indeed she has no strategy for these cases:

- For 2, Alice cannot move and thus loses.
- For 3, she cannot move after $(1, 2, 3) \mapsto (2, 2)$.
- For 4, without loss of generality Alice first moves $(1, 2, 3, 4) \mapsto (1, 3, 3) \mapsto (2, 3)$ and she cannot move further.
- For 6: Note that in $[6]$, there are exactly 2 numbers in each residue class mod 3. We make the following observations:
  - In every move we need to pick two numbers from different residue classes since if two unequal numbers are congruent mod 2 and mod 3 then they have to differ by at least 6.
  - The average of these numbers will lie in the third residue class.

  This means that at every step the parity of the three residue classes remains the same, which means that we can never achieve the state of one of the residue classes having one element left and others having zero elements.

□

Problem 3. Euclid has a tool called *splitter* which can only do the following two types of operations:

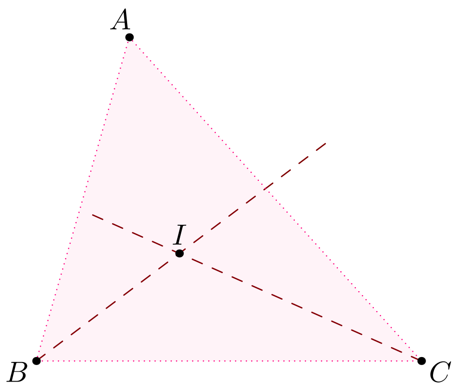
- Given three non-collinear marked points X, Y, Z, it can draw the line which forms the interior angle bisector of $\angle XYZ$.
- It can mark the intersection point of two previously drawn non-parallel lines.

Suppose Euclid is only given three non-collinear marked points A, B, C in the plane. Prove that Euclid can use the *splitter* several times to draw the centre of the circle passing through A, B, and C.

Solution. Let I be the incenter of $\triangle ABC$. We begin by proving some lemmas about the power of the tool *splitter*.

Lemma 1: Euclid can construct the incenter of any three non-collinear given points.

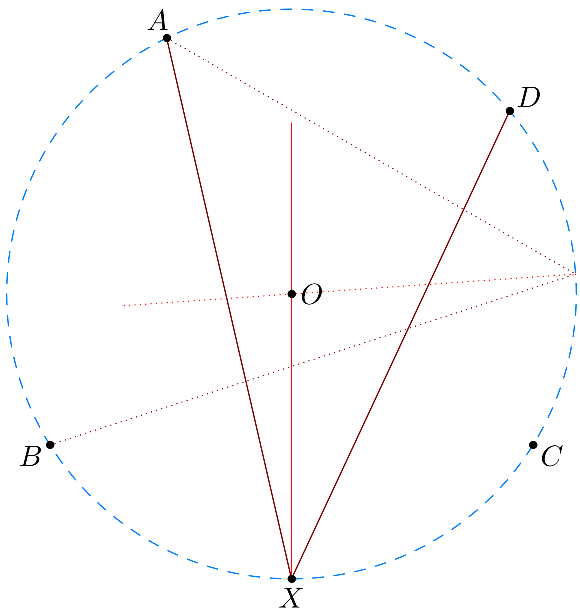
Proof. Observe that the incenter is just given by taking the intersection of any two angle bisectors.



□

Lemma 2: Given four marked points which Euclid knows to be on a circle, Euclid can draw the circumcentre of the circle through the four points.

Proof. Let the four points be A, B, C, D in this order. Now, observe that Euclid can draw the midpoint of arc $\widehat{BC}$ by intersecting the angle bisectors of $\angle BAC$ and $\angle BDC$. Let this point be called X. Thus, Euclid can draw the angle bisector of $\angle BXC$ to get the perpendicular bisector of BC. Repeating the process, Euclid can also get the perpendicular bisector of CD and intersecting these gives the circumcenter of cyclic quadrilateral $ABCD$.



□

Lemma 3. If Euclid can draw the $B-$ excenter of $\triangle BIC$, then he can also draw the circumcenter of $\triangle ABC$.

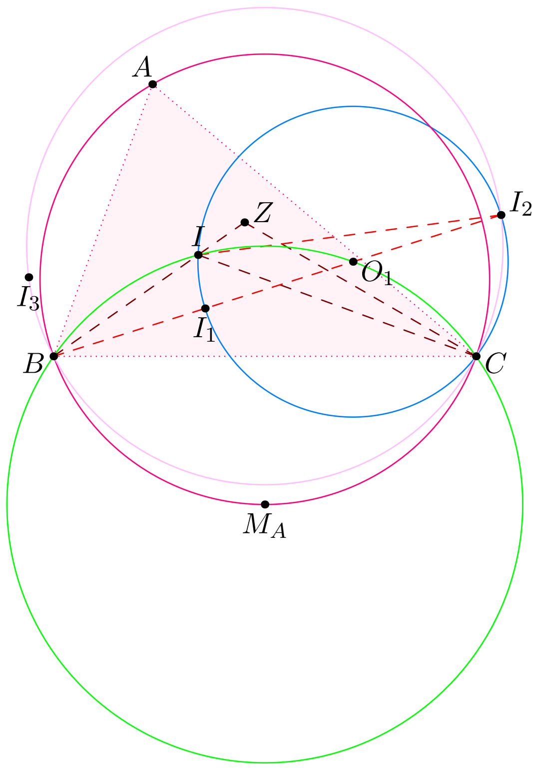
Proof. Let I_1 and I_2 be the incenter and $B-$ excenter of $\triangle BIC$. Now, since (II_1CI_2) are concyclic and Euclid can construct all of them, he can draw the circumcentre of $\triangle II_1C$. Let this be called O_1.

Observe that BIO_1C are concyclic since O_1 is the midpoint of arc $\overset{\frown}{IC}$ in (BIC). Thus, Euclid can also draw the circumcentre of $\triangle BIC$. Let this be called M_A. Now, M_A is the midpoint of arc $\overset{\frown}{BC}$ in (ABC). Thus, again using Lemma 2, we can construct the circumcentre of (ABC). □

Thus, we just need to show that Euclid can draw the external angle bisector of $\angle BIC$ since he can then intersect it with the angle bisector of $\angle IBC$ to get I_2.

This can be done if we have any point on the extension of line BI but observe that the intersection of BI with the angle bisector of $\angle ICA$ is such a point! Thus, we are done!

All the constructions are now shown in the below figure.



□

Solution 2. We provide an alternate statement similar to Lemma 3.

Lemma 3†. If Euclid can draw the B and C excenters of $\triangle BIC$ then Euclid can draw the perpendicular bisector of BC.

Proof. We will call these two points I_2 and I_3. Observe that $\angle I_3BI_2 = \angle I_3CI_2 = 90°$. Thus, I_3BI_2C is cyclic. Now, Euclid can take the intersection of angle bisectors $\angle BI_3C$ and $\angle BI_2C$ which will now intersect on the perpendicular bisector of BC. Let this be O_2. Now, Euclid can construct the angle bisector of $\angle BO_2C$ as desired! □

After this lemma, we can finish as before. □

Problem 4. Let $n \geqslant 3$ be a positive integer. Find the largest real number t_n as a function of n such that the inequality

$$\max(|a_1+a_2|, |a_2+a_3|, \ldots, |a_{n-1}+a_n|, |a_n+a_1|) \geqslant t_n \cdot \max(|a_1|, |a_2|, \cdots, |a_n|)$$

holds for all real numbers $a_1, a_2, \cdots, a_n$.

Answer. When n is even, the maximum t_n is 0 and when n is odd, the maximum t_n is $\frac{2}{n}$.

Solution. We consider the case when n is even and when n is odd separately.

- **When n is even:** let $a_i = (-1)^i$, this tells us that

  $$0 \geq t_n$$

  but we also have $t_n \geq 0$ as

  $$\max(|a_1+a_2|, |a_2+a_3|, \ldots, |a_{n-1}+a_n|, |a_n+a_1|) \geqslant 0$$

  always. Thus, when n is even, $\boxed{t_n = 0}$.

- **When n is odd:** let $a_i = (-1)^i(n-2i)$. Now, $|a_n| = n$ and for all $i \leq n-1$:

  $$|a_i + a_{i+1}| = |(-1)^i(n-2i-(n-2i-2))| = |2i-2i+2| = 2$$

  Also,

  $$|a_n + a_1| = |(-1)^n(-n) + (-1)(n-2)| = |n-n+2| = 2$$

  Thus,

  $$t_n \leq \frac{2}{n} \tag{5}$$

  Claim. $|a_1+a_2| + |a_2+a_3| + \ldots + |a_{n-1}+a_n| + |a_n+a_1| \geq 2\max(|a_1|, |a_2|, \cdots, |a_n|)$.

  Proof. Without loss of generality $|a_1| = \max(|a_1|, |a_2|, \cdots, |a_n|)$. Let $a_{n+1} = a_1$, then

  $$|a_1+a_2| + |a_2+a_3| + \ldots + |a_{n-1}+a_n| + |a_n+a_1| = \sum_{i=1}^{n} \left|(-1)^i (a_i + a_{i+1})\right|$$

  and then, by triangle inequality, we have

  $$\begin{aligned} \sum_{i=1}^{n} \left|(-1)^i (a_i + a_{i+1})\right| &\geqslant \left|\sum_{i=1}^{n} (-1)^i (a_i + a_{i+1})\right| \\ &\geqslant \left|-a_1 - a_{n+1} + \sum_{i=2}^{n} \left((-1)^{i-1} + (-1)^i\right) a_i\right| \\ &\geqslant |-a_1 - a_{n+1}| \\ &\geqslant 2|a_1| \end{aligned}$$

  Thus,

  $$|a_1+a_2| + |a_2+a_3| + \ldots + |a_{n-1}+a_n| + |a_n+a_1| \geq 2|a_1| = 2\max(|a_1|, |a_2|, \cdots, |a_n|)$$

  □

  Since,

  $$n\left(\max(|a_1+a_2|, |a_2+a_3|, \ldots, |a_n+a_1|)\right) \geqslant |a_1+a_2| + |a_2+a_3| + \ldots + |a_n+a_1|$$

  we must have that

  $$\max(|a_1+a_2|, |a_2+a_3|, \ldots, |a_{n-1}+a_n|, |a_n+a_1|) \geqslant \frac{2}{n}\max(|a_1|, |a_2|, \cdots, |a_n|) \tag{6}$$

  Combining (5) and (6), we get that for n odd, $\boxed{t_n = \frac{2}{n}}$. □

Problem 5. Greedy goblin Griphook has a regular 2000-gon, whose every vertex has a single coin. In a move, he chooses a vertex, removes one coin each from the two adjacent vertices, and adds one coin to the chosen vertex, keeping the remaining coin for himself. He can only make such a move if both adjacent vertices have at least one coin. Griphook stops only when he cannot make any more moves. What is the maximum and minimum number of coins that he could have collected?

Answer. The maximum is 1998 and the minimum is 668.

Let the vertices of the 2000-gon be $A_1, \cdots A_{2000}$. We also use indices of vertices modulo 2000 i.e. $A_0 = A_{2000}, A_1 = A_{2001}$, and so on.

Solution 1. The solution has 4 parts:

(a) Griphook can collect 1998 coins.

(b) Griphook can collect exactly 668 coins and be unable to make a move.

(c) Griphook cannot collect more than 1998 coins.

(d) Griphook could not have collected less than 668 coins when he cannot make further moves.

We prove these 4 parts in order. For convenience, we use the following notation:

$$a_1 \dots a_{2000} \mapsto a'_1 a'_2 \dots a'_{2000}$$

to denote that if Griphook has a_i coins on vertex A_i for all i then he can make a sequence of moves changing the number of coins to a'_i on vertex A_i where each $a_i, a'_i \geq 0$. We also use s^i for any string s to represent the concatenation s i times i.e. $s^i = \underbrace{ss\dots s}_{s \text{ is repeated } i \text{ times}}$.

(a) Griphook can collect 1998 coins:

Lemma 1. $1^n 01 s \mapsto 010^n s$ for any $n \geq 1$ and any string s.

Proof. Observe that $1^n 01 \mapsto 1^{n-1}010$. Repeat this move n times to get the desired result. □

Now, observe that Griphook can collect 1998 coins since:

$$1^{2000} \mapsto 1^{1996}0201 \mapsto 1^{1995}01010 \overset{\text{Lemma 1.}}{\mapsto} 010^{1996}10$$

(b) Griphook can collect exactly 668 coins and be unable to make a move:
Observe that Griphook can collect 668 coins in the following way:

$$1^{2000} \mapsto 1(020)^{666}1 = 1020(020)^{664}0201 \mapsto 0110(020)^{664}0110$$

and he cannot make any further moves. □

(c) Griphook cannot collect more than 1998 coins:
Let o be the number of coins on $A_o = \{A_1, A_3, \dots, A_{1999}\}$ and e be the number of coins on $A_e = \{A_2, A_4, \dots, A_{2000}\}$. Now, observe that in any move

$$o - e \pmod 3$$

is unchanged. But, it is initally 0 and thus, since atleast one coin is remaining at the end, we must have $|o|, |e| \geq 1$ and thus atleast 2 coins remain at the end.

(d) Griphook could not have collected less than 668 coins when he cannot make further moves:

We need to show that at least 668 moves are possible. Split the vertices into two disjoint regular 1000-gons consisting of the vertices at the even positions and odd positions respectively. Call these polygons Polygon A = $\{v_1, v_3, \ldots, v_{1999}\}$ and Polygon B = $\{v_0, v_2, \ldots, v_{1998}\}$. We will consider the indices modulo 2000, i.e., $v_{2000} = v_0$, $v_{-1} = v_{1999}$, and so on. In any move, Griphook is choosing a vertex v_{i+1}, such that are both v_i and v_{i+2} are non-empty and removing a coin from each (and adding a coin to v_{i+1}). Suppose Griphook made ≤ 333 moves on vertices of Polygon B. Then there must be three consecutive vertices $\{v_{j-2}, v_j, v_{j+2}\}$ in Polygon B that Griphook never makes moves on. This means that the coins that were at v_{j-1} and v_{j+1} initially are never removed. Hence, Griphook needs at least 334 moves on Polygon B and similarly at least 334 moves on Polygon A to ensure that he cannot make any further moves, for a total of at least 668 moves. □

Solution 2. We provide an alternate proof of (d). Let $c(A_i)$ be the number of coins at vertex A_i.

Lemma 1. $c(A_i) + c(A_{i+1}) \leqslant 2$ for all i at every stage during the process.

Proof. Observe that no move can increase the quantity $c(A_i) + c(A_{i+1})$ and initially, we have $c(A_i) + c(A_{i+1}) = 2$. □

Now, let the final position have a_i coins on vertex A_i. Now, we would like to prove that $\sum a_i \leq 1332$ since $2000 - \sum a_i$ is the number of coins that Griphook picked up.

Lemma 2. Let $a_1, a_2, \ldots a_n$ be any non-negative integers such that:

- $a_i + a_{i+1} \leq 2$ for all i,
- and whenever a_i is positive then $a_{i-2} = a_{i+2} = 0$.

where indices are modulo n. Then,

$$\sum_{i=1}^{n} a_i \leq 2\left\lfloor \frac{n}{3} \right\rfloor$$

Proof. We will induct on this claim. Observe that when $n = 1$ or $n = 2$, the claim is immediate as $a_i = a_{i+2}$ so none of the values can be > 0. If $n = 3$, then no two consecutive values can be > 0 and thus $\sum a_i \leq 2$ as required. Now if $n > 3$, we proceed by induction.

If there exists an i such that a_i and a_{i+1} are positive then

$$a_i = a_{i+1} = 1, \qquad a_{i+2} = a_{i+3} = a_{i-1} + a_{i-2} = 0.$$

Thus, we can delete a_i, a_{i+1} and a_{i-1} to be left with $n - 3$ numbers which also satisfy the properties given and we are done by the inductive hypothesis.

If there is no such i and if a_i is positive, then $a_{i-1} = a_{i+1} = a_{i+2} = a_{i-2} = 0$. Hence, we can delete a_{i-1}, a_i, a_{i+1} and the requisite properties are still satisfied by the remaining numbers. Thus, we are done! □

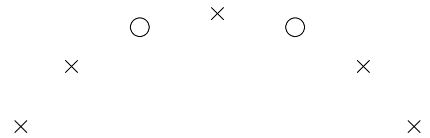
Plugging in $n = 2000$ gives us the desired result. □

Solution 3. We provide an alternate proof of (c). This proof generalizes to show that **for every $n > 5$, if we start with a regular n-gon with a coin on every vertex, Griphook can take at most $n - 2$ coins**. (The appropriate change of 2000 to n in each step is left to the reader)

If in a move, the number of coins on a vertex is increased, then the number of coins on both the adjacent vertices are reduced. Therefore, the sum of number of coins on any two adjacent vertices never increases throughout the process. In particular, this sum is always ≤ 2. Similarly, we can show that the sum of the number of coins in three adjacent vertices is ≤ 3.

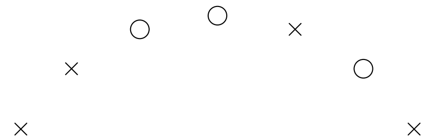
Let us assume that 1999 moves are possible. Then the final state of the board is: a single coin on a vertex, and every other vertex being empty. We consider the moves in reverse, i.e., a move consisting of removing a coin from a vertex and placing two coins in the adjacent vertices. Then, by making this“reverse move” 1999 times, we should be able to go from this final state to the state where every vertex has a single coin.

After the first reverse move, the board is in the state

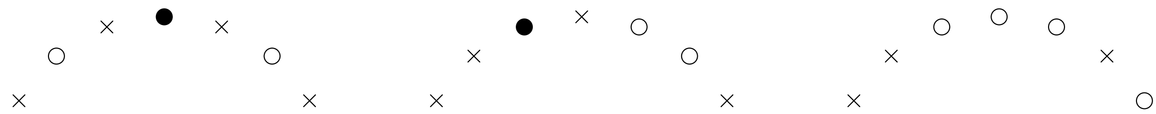


× *means empty vertex,* ∘ *means vertex with one coin,* • *means vertex with two coins.*

Now, the next reverse move puts the board in the state (due to symmetry)



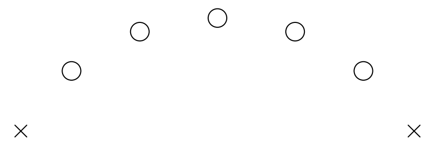
For the next reverse move, we have 3 choices, which creates the possible states



I **II** **III**

Case I

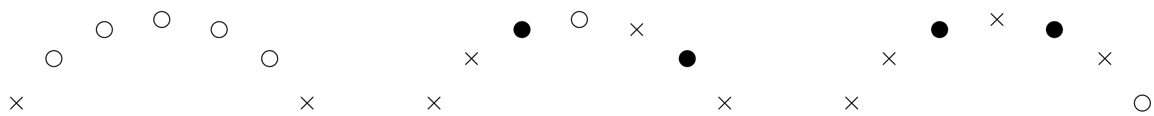
Now, if the next reverse move is performed on one of the vertices with the single coins, then we create the situation where a vertex with two coins is next to a vertex with one coin. This is a contradiction to our initial observation that the sum of number of coins on any two adjacent vertices is always ≤ 2. Therefore, the next reverse move must be performed on the vertex with two coins on it. This creates the state



Now, any reverse move necessarily creates a vertex with two coins next to a vertex with one coin. Therefore, we are unable to perform any further reverse moves in case I.

Case II

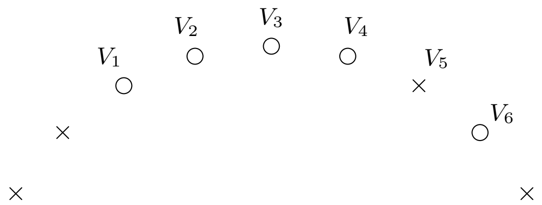
There are three possible choices for the next reverse move, creating the possible states



The first state has already been ruled out in case I. The second one violates the rule of number of coins in two adjacent vertices being ≤ 2. The third state violates the rule of number of coins in two adjacent vertices being ≤ 3.

Therefore we see that the cases I and II are not possible. So, we third reverse move must land us in case III.

Now, let us consider the fourth reverse move. The only reverse move that does not create a $2-1$, or a $2-0-2$, thereby violating our initial rules, is the move on the isolated vertex with a coin, creating the state



From here on, we claim that, for all $4 \leq n \leq 1998$, after the n-th reverse move, the state of the polygon must be the following: a coin each on vertices $V_1, V_2, \ldots, V_n$, and V_{n+2}, and rest of the vertices being empty.

This claim can be proven by induction on n. The base case for $n = 4$ has already been shown. Let us assume the result to hold for n. If the next reverse move is made on V_i where $i = 1$ or 2, then V_{i+1} has 2 coins and V_{i+2} holds one coin, which is a contradiction. If the reverse move is made on V_i where $3 \leq i \leq n$, then V_{i-1} has 2 coins and V_{i-2} holds one coin, which is a contradiction as well. So, the only possible reverse move is on V_{n+2}. This leads to the state where vertices $V_1, V_2, \ldots, V_{n+1}$ and V_{n+3} hold a coin each, and rest of the vertices are empty. This completes the induction step.

So, after 1998 reverse moves, every vertex except V_{1999} hold a coin each. Now, we can clearly see that no reverse move exists that can take this state to the state with a coin on all 2000 vertices.

Problem 6. Let $b \geqslant 2$ be a positive integer. Anu has an infinite collection of notes with exactly $b-1$ copies of a note worth $b^k - 1$ rupees, for every integer $k \geqslant 1$. A positive integer n is called *payable* if Anu can pay exactly n^2+1 rupees by using some collection of her notes. Prove that if there is a payable number, there are infinitely many payable numbers.

Let $s_b(m)$ denote the sum of digits of m in base b. Note that n is payable if and only if $n^2 + 1 = m - s_b(m)$ for some $m > 0$. We need to show that if there is a payable positive integer n then there are infinitely many payable positive integers n. We begin by proving that the following is an equivalent statement:

> Let $b \geq 2$ be a positive integer. Suppose there exists a positive integer n_0 such that $n_0 - s_b(n_0) - 1$ is a perfect square. Prove that there are infinitely many positive integers n such that $n - s_b(n) - 1$ is a perfect square.

To show this, it suffices to show that given any $t \in \mathbb{N}$, $x - s_b(x) = t$ has only finitely many solutions $x \in \mathbb{N}$. Let z be the number whose base b representation is obtained by deleting the last digit of the base b representation of x, i.e., $z = \dfrac{x - r(x)}{b}$, where $r(x)$ is the remainder when x is divided by b. Then $x - s_b(x) = bz - s_b(z)$. Now, note that $bz - s_b(z)$ is a strictly increasing sequence, since $b(z+1) - s_b(z+1) \geq bz + b - s_b(z) - 1 > bz - s_b(z)$. Therefore, we can see that for a fixed t, there are exactly 0 or b many solutions to the equation $x - s_b(x) = t$.

We will now work with the restatement. For simplicity, we will separately deal with the cases when $b = 2$ and $b > 2$.

Case 1: $\mathbf{b = 2}$:

For any integer $k \geq 2$, take $n = 2^{2k} + 2^{k+1} + 4 + 2$. Clearly $s_2(n) = 4$, so

$$n - s_2(n) - 1 = 2^{2k} + 2^{k+1} + 1 = (2^k + 1)^2$$

is a perfect square.

Also, for any integer $k \geq 3$, we can take $n = 2^{2k} + 2^{k+2} + 8$. Clearly, $s_2(n) = 3$, so

$$n - s_2(n) - 1 = 2^{2k} + 2^{k+2} + 4 = (2^k + 2)^2$$

Case 2: $\mathbf{b > 2}$: Suppose for some integers $n_0 > 0$ and $x \geqslant 0$, we have $n_0 - s_b(n_0) - 1 = x^2$.

Solution 1. Observe that

$$b - 1 \mid n_0 - s_b(n_0) = x^2 + 1.$$

This means -1 is a quadratic residue modulo $b-1$, and that is all we'll use. Let r be a residue modulo $b-1$ satisfying $r^2 + 1 \equiv 0 \pmod{b-1}$; so $0 \leq r \leq b-2$.
We must have $r > 0$. Then, by replacing r by $b-1-r$ if needed, we can assume $r \leq \frac{b-1}{2}$.
Let

$$a = \frac{r^2+1}{b-1} < \frac{(b-1)^2+1}{b-1} = b - 1 + \frac{1}{b-1} < b.$$

For any positive integer $k \geq 2$, consider $n = b^{2k} + 2(r-1)b^k + ab$. Since $0 \leq a < b$ and $0 \leq 2(r-1) < b$, we have $s_b(n) = 1 + 2(r-1) + a$. Hence

$$n - s_b(n) - 1 = b^{2k} + 2(r-1)b^k + ab - a - 2r + 1 - 1 = b^{2k} + 2(r-1)b^k + r^2 - 2r + 1 = (b^k + r - 1)^2$$

is a perfect square. $\square$

Solution 2. Let α be any integer such that $s_b(\alpha^2) = n_0 - s_b(n_0) - 1 = x^2$. Now, consider $n = \alpha^2 \cdot b^{2N} + n_0$ for any N such that $b^{2N} > n_0$. Thus,

$$n - s_b(n) - 1 = \alpha^2 \cdot b^{2N} - s_b(\alpha^2) + x^2 = (\alpha b^N)^2$$

Now, we just need to show the existence of such an α. Firstly, observe that since

$$b-1|x^2+1,$$

we can assume $x>0$.

Now, let

$$\alpha=\sum_{i=0}^{x-1} b^{2^i}$$

Now, observe that

$$\alpha^2=\sum_{i=1}^{x} b^{2^i}+\sum_{0\le i<j\le x-1} 2b^{2^i+2^j}$$

Observe that any term of the form $k=2^i+2^j$ with $i<j$ uniquely determines i,j and cannot be a power of 2. Thus, $s_b(\alpha^2)=x^2$ as desired. □

Remark. Solution 2 works for base 2 too since $s_b((2^k-1)^2)=k$ for any positive integer k which shows that the required α exists. (Note that $x>0$ is given in the original problem statement so we do not need it)
