Indian National Astronomy Olympiad (INAO) - 2025

Jointly organized by
Homi Bhabha Centre for Science Education (HBCSE-TIFR)
and
Indian Institute of Science Education and Research Mohali (IISER-Mohali)

Provisional Solutions

Roll Number: ☐☐☐☐ ☐☐☐☐ ☐☐☐☐ **Date:** 01 February 2025

Duration: Three Hours **Maximum Marks:** 100

Please Note:

- Write your **Roll Number** in the boxes above.
- There are a total of **10 pages and 12 questions** in this booklet. Maximum marks for each sub-question are indicated.
- For all questions, the *process* involved in arriving at the solution *is equally important* as the final answer. Valid assumptions and approximations are perfectly acceptable. Please write your method clearly, explicitly stating all reasoning / assumptions / approximations.
- Use of non-programmable scientific calculators is allowed.
- **The answer-sheet must be returned to the invigilator**. You can take this question paper back with you.

Useful Constants

Quantity	Symbol	Value
Mass of the Sun	$M_\odot$	1.989×10^{30} kg
Mass of the Earth	$M_\oplus$	5.972×10^{24} kg
Mass of the Moon	M_c	7.347×10^{22} kg
Radius of the Sun	$R_\odot$	6.955×10^{8} m
Radius of the Earth	$R_\oplus$	6.371×10^{6} m
Radius of the Moon	R_c	1.737×10^{6} m
Speed of Light	c	2.998×10^{8} m/s
Astronomical Unit	$a_\oplus \equiv 1$ A.U.	1.496×10^{11} m
Sun-Venus distance	0.72 A.U.	1.077×10^{11} m
Solar Constant (at Earth)	S	1366 W/m^2
Gravitational Constant	G	6.674×10^{-11} N m^2/kg^2
Planck Constant	h	6.626×10^{-34} J/Hz

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Question 1

Consider a telescope with a primary (convex lens) with a diameter of 5 cm and a focal length of 60 cm, and a secondary (convex lens) with a focal length of 3 cm. This telescope is used to project a perfectly focused image of the Sun at a distance of 60 cm from the secondary. You are given that the diameter of the Sun subtends an angle of $30'$ in the sky.

(a) What is the distance between the primary and secondary? [2]

(b) Draw a ray diagram marking all the optical elements, images and distances. [3]

(c) What is the size of the image? [3]

(d) What is the physical size of the smallest sunspot that can be resolved by this arrangement? [2]

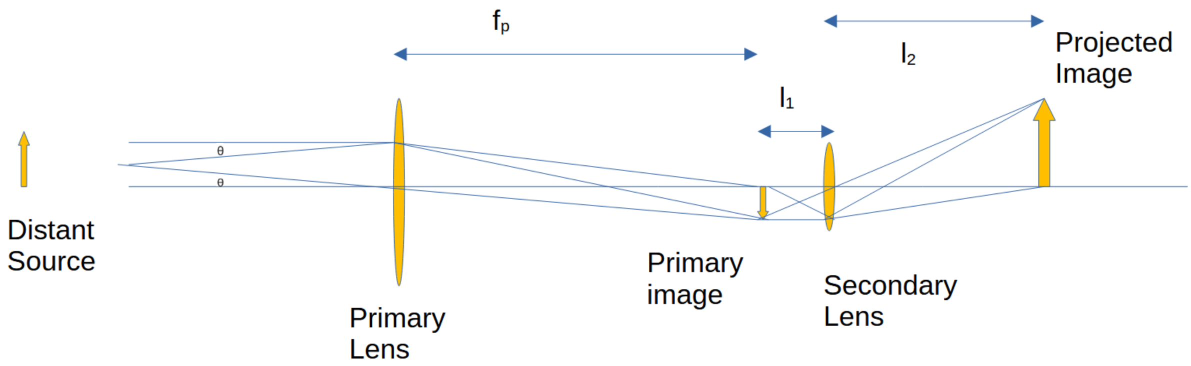
Solution: As we are given that a perfectly focussed image is projected on a screen located at a finite distance, it has to be a real image. From here it also follows that the distance between the primary focus and the secondary lens must be more than the focal length of the secondary. As we are dealing with only refraction here, the image has to form behind the secondary.

(a) If the focal length of the primary lens is f_p, that of the secondary lens is f_s and the image is at a distance l_2 from the secondary lens then the distance between the focus of the primary and the secondary lens is:

$$l_1 = \left(\frac{1}{f_s} - \frac{1}{l_2}\right)^{-1} = 3.16 \text{ cm}.$$

Distance between the primary and secondary is $f_p + l_1 = 60 + 3.16 = 63.16$ cm.

(b) The illustrative ray diagram for the setting is given below. Note that the finite size of the source and the image is not shown in the ray diagram.:



(c) The size of the primary image is the product of the angle subtended by the Sun and the focal length of the primary lens ($\theta \times f_p$). Size of the projected image is then $\theta f_p l_2/l_1$. The size of the image is:

$$I = \theta \left(l_2 \frac{f_p}{f_s} - f_p \right),$$

where θ is the angle subtended by the Sun. The angle subtended by the diameter of the Sun is $30' = 0.00873$ radians. Using given numbers, we have $I = 9.95$ cm.

(d) Resolution is determined from the diameter D of the primary lens: $\delta\theta = 1.22\ \lambda/D$. Assuming observation is at a wavelength $\lambda = 5000\ \mathring{A}$, we have $\delta\theta = 1.22 \times 10^{-5}$ radians or $2.6''$. This corresponds to a physical scale $\delta\theta \times 1$ A.U. $= 1825$ km. Numerical solution with a different choice of wavelength within the optical window is acceptable.

Question 2

An isolated star outside the solar system normally moves with a certain constant velocity with respect to the Sun. However, if there is a planet in orbit around it, there will be an appropriate periodic change in its motion. This variation in motion can now be detected using sensitive measurements of the Doppler shift of spectral lines. However, only the line-of-sight component of the total velocity of the star can be detected using this method. This motion, combined with the time period of the orbital motion, can be used to estimate the mass of a planet orbiting a distant star.

Suppose that a star of mass m_s has a planet of mass m_p in a circular orbit. The separation between the objects is r, and circular speed of the planet is v_p.

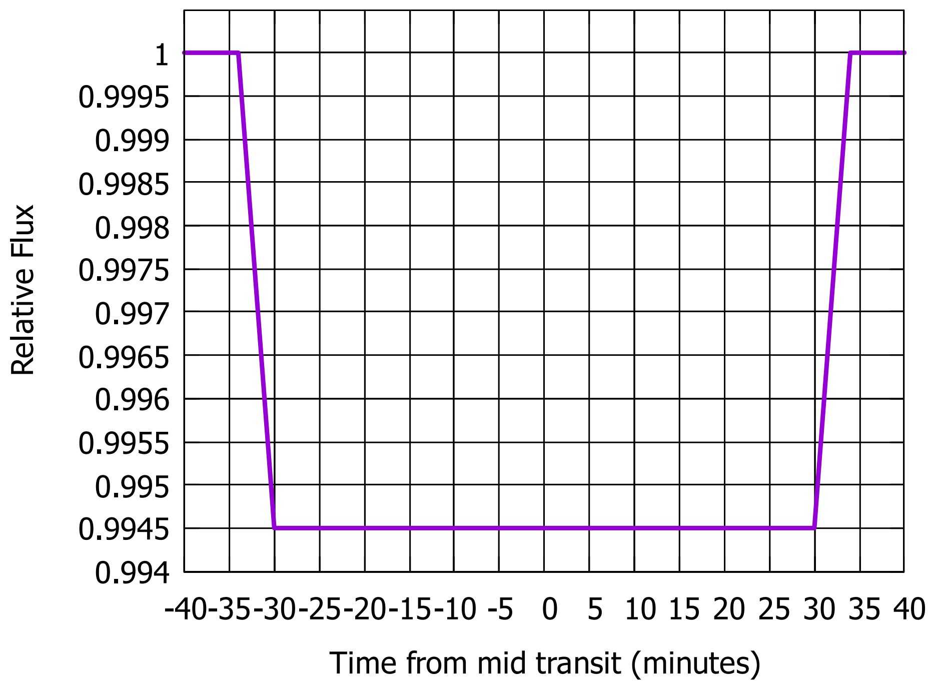


Figure 1: Light curve of the Planet.

(a) Assuming that $m_p \ll m_s$, find an expression for m_p in terms of observables (orbital time period P, v_s : the periodic variation in the radial velocity of the star, and mass of the star m_s that can be estimated from other considerations). [3]

(b) We can determine the time period P from the periodicity of variation of Doppler shift in the spectral lines and estimate m_s from the spectral type of the star. A special case is where the planet is transiting in front of (i.e. partially eclipsing) the star as it orbits. In this case the inclination is known to be 90°. This leads to the light curve of the type shown in Figure 1. Compute the orbital radius of the orbit of the planet around the star. State your approach and reasoning clearly. You are provided that $P = 6.1$ days, $m_s = 0.0898\ M_\odot$ and $v_s = 1.353$ m/s. Compute the mass and the radius of the planet in the system for which the light curve has been given in Figure 1. [5]

(c) Albedo is defined as the reflectivity of a planet. Earth has an Albedo of 0.31, for instance. Assume the planet under consideration has a similar Albedo to that of the Earth. Determine the average surface temperature of the planet, and thus whether water can be found on it in the liquid state, an important condition for a planet to be habitable. The radius and the temperature of the star are 0.1192 $R_\odot$ and $T_s = 2566$ K respectively. [5]

(d) Planets with average density above 3 g cm^{-3} can be classified as rocky while those below this density can be dubbed as gaseous planet. Find out if the planet is rocky or gaseous. [2]

Solution:

(a) The acceleration of an object in a circular orbit or radius r at constant speed v is v^2/r. We normally apply this to the planet, but it is also applicable to the star as it also circles the center of mass of the system. Thus

$$\frac{m_s v_s^2}{r_s} = \frac{G m_s m_p}{(r_s + r_p)^2}$$

$$v_s^2 = \frac{G m_p}{r_s(1 + r_p/r_s)^2} = \frac{2\pi G m_p}{P v_s (1 + m_s/m_p)^2}$$

Using $m_p \ll m_s$, we get

$$m_p = \left(\frac{P}{2\pi G}\right)^{1/3} m_s^{2/3}\, v_s.$$

(b) Radius of the planet's orbit:

$$\frac{r_P}{1\ AU} = \left(\frac{P}{365}\right)^{2/3} \left(\frac{M_s}{M_\odot}\right)^{1/3}$$

Therefore, $r_P = 0.02925$ A.U.

Since the planet is transiting, $i \approx \pi/2$, and thus we get m_p completely. The duration of the downward-sloping (or upward-sloping) part of the dip gives us a time interval during which the the planet is effectively moving at $v_p = v_s m_s/m_p$. This gives us the diameter of the planet.

Calculation: Mass of the planet:

$$m_p = 4.56 \times 10^{24}\ \text{kg}$$

Therefore, $v_p = v_s m_s/m_p$ implies

$$v_p \approx 52\ \text{km s}^{-1}$$

Since $v_s \ll v_p$, the diameter of the planet can be obtained from its velocity and the duration of slope (duration of downward/upward sloping part is approximately 4m):

$$D = t \times v_p = 4 \times 60 \times 52 = 12,480\ \text{km}$$

Alternate Solution 1

Ratio of fluxes with and without eclipsing:

$$\frac{F_{eclipsing}}{F_s} = 1 - \frac{R_P^2}{R_s^2}$$

and therefore

$$R_P \approx 6148 \text{ km}$$

Alternate Solution 2

$$2(R_s - R_p) = v\ t$$

Sustituting $R_s = \frac{R_p}{\sqrt{0.0055}}$ gives

$$R_p = 7483 \text{ km}$$

Subsequent answers for each of the alternative solutions are accepted as correct.

(c) Luminosity of the star

$$\sigma T_s^4 4\pi R_s^2$$

Flux at the planet:

$$\frac{\sigma T_s^4 4\pi R_s^2}{4\pi r_p^2}$$

Power absorbed:

$$(1 - A)\frac{\sigma T_s^4 R_s^2}{r_p^2}\pi R_P^2$$

Power radiated by the Planet:

$$\sigma T_P^4 4\pi R_P^2$$

For maintaining the temperature:

$$(1 - A)\frac{\sigma T_s^4 R_s^2}{r_p^2}\pi R_P^2 = \sigma T_P^4 4\pi R_P^2$$

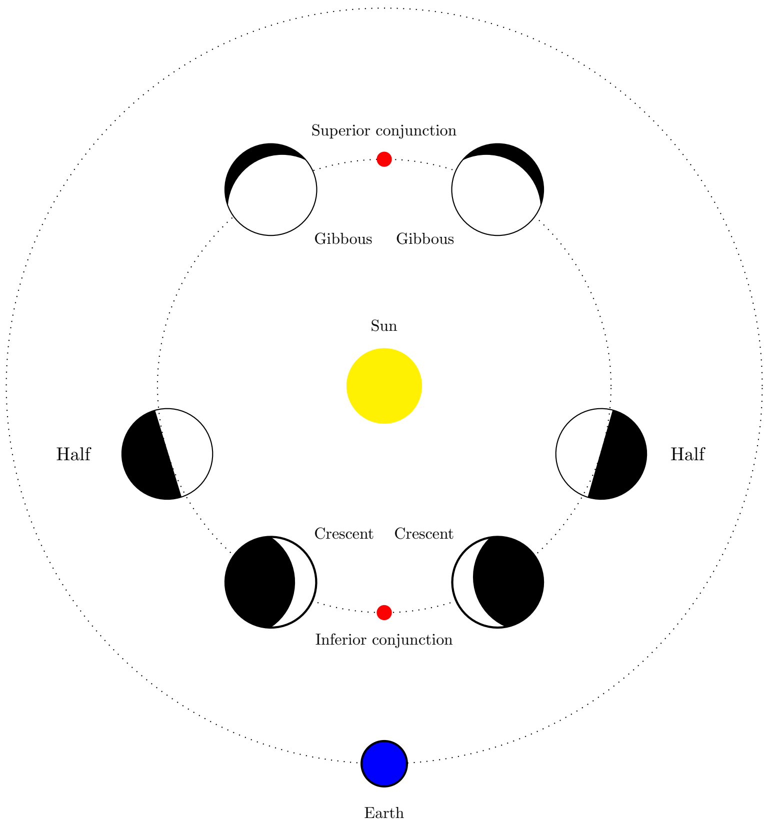
$$r_p = \frac{(1 - A)^{1/2}}{2}\left(\frac{T_s}{T_P}\right)^2 R_s$$

For the distance of the planet from the star $r_p = 0.02925$ A.U., the temperature of the planet can be obtained to be approximately 227.52 K. The planet can support frozen water.

(d) Given the mass and radius, we can show that the density is about 4.48 g cm^{-3}, which makes it a rocky planet.

Using alternate solution 2, the density is 2.6 g.cm^{-3}. In this case the correct solution is a gas giant.

Question 3



Phases of Venus

(Figures not to scale)

The flux of light reaching the Earth from Venus varies due to (i) the distance of Venus from Sun, (ii) the distance of Venus from the observer on the Earth, (iii) the amount of illuminated surface visible to the observer and (iv) the reflective power of the Venus surface and its rotational speed (see Figure below).

(a) Set up the equation for flux of light reaching the Earth after reflection from Venus. Assume the Sun-Venus and the Sun-Earth orbit to be circular in the same plane with the radii r and Δ respectively; the radius of Venus to be a, the Earth-Venus distance to be ρ and the Earth-Venus-Sun angle to be ψ. [5]

(b) Find the value of ρ (in terms of r and Δ) so that Venus has maximum flux at that distance. [5]

(c) Estimate the portion of the Venus disk that is illuminated at its maximum flux. Is it in Crescent or Gibbous phase? [2]

(d) The flux received from Venus varies by a factor of 3 between extremes during its entire synodic cycle. How do you explain this almost constant apparent flux in spite of large variation in distance and illumination? [3]

Solution:
(a) The illuminated area of disk of the Venus is minimum near the crescent of Venus and maximum near the point of opposition. Hence $A = \frac{1}{2}\pi a^2(1 + \cos\psi)$.

The flux of light reaching the Earth after reflection from Venus is

$$F = kA\frac{1}{r^2}\frac{1}{\rho^2}$$

$$F = k\frac{1}{2}\pi a^2(1 + \cos\psi)\frac{1}{r^2}\frac{1}{\rho^2}$$

In the Sun, Venus, Earth triangle, using geometry we have,

$$\Delta^2 = \rho^2 + r^2 - 2r\rho\cos\psi$$

Putting the value of $\cos\psi$ from this expression into the flux equation, we get

$$F = k\frac{1}{2}\pi a^2(1 + \frac{\rho^2 + r^2 - \Delta^2}{2r\rho})\frac{1}{r^2}\frac{1}{\rho^2}$$

which simplifies to

$$F = k\frac{1}{4}\pi a^2(2r\rho + \rho^2 + r^2 - \Delta^2)\frac{1}{r^3}\frac{1}{\rho^3}$$

(b) To find the particular value of ρ, we need to differentiate the flux F, w.r.t. ρ and then put the expression equal to zero.

$$\frac{dF}{d\rho} = k\frac{1}{4}\pi a^2 r^{-3}\rho^{-6}\rho^2(-4r\rho - \rho^2 - 3r^2 + 3\Delta^2) = k\frac{1}{4}\pi a^2 r^{-3}\rho^{-4}(-4r\rho - \rho^2 - 3r^2 + 3\Delta^2)$$

Hence the condition for maximum flux is

$$4r\rho + \rho^2 + 3r^2 = 3\Delta^2$$

Taking the unit of r, ρ and Δ in terms of Astronomical Unit (A.U.), and putting the relevant values of r and Δ we get the quadratic expression for ρ from the last equation as

$$\rho^2 + 2.88\rho + 3 \times 0.72 \times 0.72 - 3 = 0$$

which has roots of the equation as $\rho_1 = 0.436$ and $\rho_2 = -3.316$. We ignore the negative root.
(c) Putting the values of r, ρ and Δ we obtain the value of $\cos\psi = -0.465$ which in turn provides us the illuminated area as $A = 0.267\pi a^2$. Hence at maximum flux Venus appears as a crescent about like the moon when it is between 4 and 5 days old.

(d) As the phase angle of Venus changes, so does its distance to the Earth. When it is closest to us, the smallest fraction of the disk is illuminated, and when it is farthest, nearly the whole disk is illuminated. This cancellation results in a compression of the range of apparent fluxes throughout the Earth-Venus synodic cycle.

Question 4

In a planetary system similar to ours (see Figure 2),

- A planet P has a moon M, and the P-M system orbits their star S.
- The orbit of M around P and that of the P-M system around S are circular, coplanar and in the same sense of rotation.
- The radius of P has been measured to be 6000 km.
- Laser ranging has determined the distance between P and M to be 400,000 km and that between S and the P-M system to be 150×10^6 km respectively.
- The angular size of S viewed from the centre of mass of the P-M system and that of M viewed from P are $30'$ and $31'$ respectively.
- M completes one orbit around P in 30 Earth days and orbital period of P around S is 360 Earth days.
- Planet P spins around its axis once in 25 Earth hours. The spin of M is synchronised with its orbit around P. The sense of rotation of P and M and of their revolution in their respective orbits are the same.

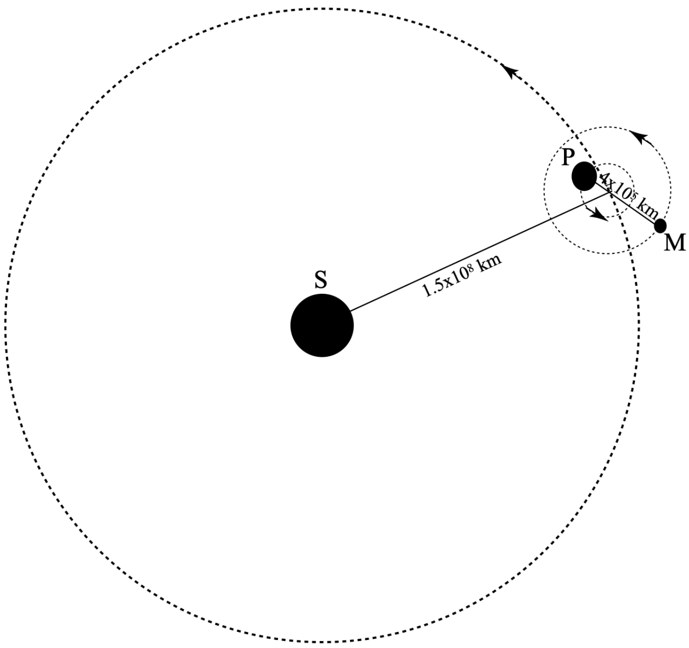


Figure 2: A schematic sketch of the Star S - Planet P - Moon M system

Given the above:

(a) Find the mass of S. [2]

(b) Assuming P and M are made of the same material, and their density is uniform, find the masses of P and M. [3]

(c) Are all the eclipses of S by M as viewed from the equator of P Total? If so, what is the duration of totality? [3]

(d) Fossil records suggest that 1×10^9 Earth years ago the spin period of P was 22 Earth hours. Tidal interaction has caused angular momentum to be transferred from the spin of P to the P-M system. What was the orbital period of M around P, 1×10^9 years ago? [3]

(e) If tidal angular momentum transfer continues at the same rate, when will the eclipses of S by M as viewed from the equator of P cease to be total? What will be the orbital period of M and spin period of P then? [4]

Solution:

1. Let $M_{\rm S}$, $M_{\rm P}$ and $M_{\rm M}$ be the masses of S, P and M respectively. The mass of the P-M system is then $M_{\rm P} + M_{\rm M}$. The orbital radius of P-M system around S is $a_{\rm SP} = 1.5 \times 10^{11}$ m and the orbital period is $T_{\rm SP} = 3.1104 \times 10^7$ s. Similarly, in the P-M system $a_{\rm PM} = 4 \times 10^8$ m and $T_{\rm PM} = 2.592 \times 10^6$ s.

   By Kepler's third law
   $$M_{\rm S} + (M_{\rm P} + M_{\rm M}) = \frac{4\pi^2 a_{\rm SP}^3}{G T_{\rm SP}^2}$$
   and
   $$M_{\rm P} + M_{\rm M} = \frac{4\pi^2 a_{\rm PM}^3}{G T_{\rm PM}^2}$$
   Thus
   $$M_{\rm S} = \frac{4\pi^2}{G}\left[\frac{a_{\rm SP}^3}{T_{\rm SP}^2} - \frac{a_{\rm PM}^3}{T_{\rm PM}^2}\right]$$
   Inserting the given values on the RHS, and using $G = 6.67 \times 10^{-11}\ {\rm m^3 s^{-2} kg^{-1}}$,
   $$M_{\rm S} = 2.0648 \times 10^{30}\ {\rm kg}$$

2. Angular size of M viewed from P is θ =31 arcmin = 0.0090175 radian. The diameter of M is then $a_{PM}\theta$ =3607 km, i.e. its radius is 1803.5 km. Thus $M_{\rm M}/M_{\rm P}$ =the volume ratio $(1803.5/6000)^3$=0.027158. So
   $$M_{\rm P} + M_{\rm M} = 1.027158 M_{\rm P} = \frac{4\pi^2 a_{\rm PM}^3}{G T_{\rm PM}^2} = 5.638244 \times 10^{24}\ {\rm kg}$$
   Thus
   $$M_{\rm P} = 5.489 \times 10^{24}\ {\rm kg}\ ;\ M_{\rm M} = 1.491 \times 10^{23}\ {\rm kg}$$

3. The said eclipses are caused by M eclipsing S. The angular size of M as viewed from P is 31 arcmin. The angular size of S should vary slightly because of the relative position of P in the orbit of the P-M system. The distance of the centre of mass from P is $(0.027158/1.027158) \times 4 \times 10^8$ m = 1.0576×10^7 m. As a fraction of $a_{\rm SP}$, this is of magnitude $\sim 7 \times 10^{-5}$, i.e. the angular size of S seen from P remains within ± 0.13 arcsec of 30 arcmin. We will ignore this small variation.

   As the angular size of M is always larger than that of S as seen from P, all the eclipses will be total. The duration of totality will be the time it takes for the relative movement of S and M, as seen from P, by (31-30) arcmin = 1 arcmin. The relative angular frequency in degrees per Earth day is

$$360\left(\frac{1}{30}-\frac{1}{360}\right)=11$$

Which gives the time for relative movement of 1 arcmin as 2.18 Earth minutes, which would be the duration of totality in an eclipse.

Optional: One may include the effect of the parallax of M from P due to the spin of the planet. The correction to the apparent differential angular speed is $-(\omega_{\rm P}R_{\rm P}/a_{\rm PM})\cos H$ where $\omega_{\rm P}$ is the spin angular speed of P and H is the hour angle at which the eclipse event is observed, giving net time for totality $2.18/(1-0.47\cos H)$ earth min, which has a value between 2.18 and 4.12 earth minutes. As the event hour angle is not being computed, this correction is not mandatory.

4. The moment of inertia of P is

$$I_{\rm P}=\frac{2}{5}M_{\rm P}R_{\rm P}^2=7.904\times10^{37}\ \text{kg m}^2$$

where $R_{\rm P}=6000$ km is its radius. The reduction in spin angular momentum over one billion years has been

$$I_{\rm P}(\omega_0-\omega)=\frac{2\pi}{3600}\left(\frac{1}{22}-\frac{1}{25}\right)7.904\times10^{37}\ \text{kg m}^2\text{s}^{-1}=7.525\times10^{32}\ \text{kg m}^2\text{s}^{-1}$$

which has increased the orbital angular momentum by the same amount. With reduced mass

$$\mu_{\rm PM}=\frac{M_{\rm P}M_{\rm M}}{M_{\rm P}+M_{\rm M}}=1.4516\times10^{26}\ \text{g}$$

One may write the orbital angular momentum as

$$L_{\rm PM}=M_{\rm P}M_{\rm M}\sqrt{\frac{Ga_{\rm PM}}{M_{\rm P}+M_{\rm M}}}=5.6298\times10^{34}\ \text{kg m}^2\text{s}^{-1}\sqrt{\frac{a_{\rm PM}}{4\times10^8\ \text{m}}}$$

One billion years ago the orbital angular momentum was $5.6298\times10^{34}-7.525\times10^{32}=5.5546\times10^{34}$ kg m^{2}s^{-1}.

Now since $a_{\rm PM}\propto L_{\rm PM}^2$ and $T_{\rm PM}\propto a_{\rm PM}^{3/2}$,

$$\frac{T_{\rm PM,then}}{T_{\rm PM,now}}=\left(\frac{L_{\rm PM,then}}{L_{\rm PM,now}}\right)^3=\left(\frac{5.5546}{5.6298}\right)^3=(0.9866)^3=0.96$$

Thus the orbital period of M around P one billion years ago was $0.96\times30=28.8$ days.

5. For total eclipses to disappear, the angular size of M as seen from P should drop below 30 arcmin. This requires an increase in $a_{\rm PM}$ to $a_{\rm PM,future}$ such that

$$\frac{a_{\rm PM,future}}{a_{\rm PM,present}}=\frac{31}{30}=1.033333$$

which will mean

$$\frac{L_{\rm PM,future}}{L_{\rm PM,present}}=\sqrt{\frac{a_{\rm PM,future}}{a_{\rm PM,present}}}=1.01653$$

The net change in angular momentum required is then $0.01653 L_{\text{PM,present}} = 9.306 \times 10^{32}$ kg m^{2}s^{-1}. At the current rate of tidal transfer of angular momentum, this will take

$$\frac{9.306 \times 10^{32}}{7.525 \times 10^{32}} \text{ billion years} = 1.24 \text{ billion years}$$

This also implies that

$$\frac{T_{\text{PM,future}}}{T_{\text{PM,present}}} = \left(\frac{L_{\text{PM,future}}}{L_{\text{PM,present}}}\right)^3 = 1.05041$$

i.e.

$$T_{\text{PM,future}} = 1.05041 \times 30 \text{ days} = 31.51 \text{ earth days}$$

The reduction in spin frequency of P is

$$\Delta\omega = \frac{\Delta L_{\text{PM}}}{I_{\text{P}}} = \frac{9.306 \times 10^{39}}{7.904 \times 10^{44}} \text{ s}^{-1} = 1.1774 \times 10^{-5} \text{s}^{-1}$$

giving the net spin angular frequency

$$\left(\frac{2\pi}{25 \times 3600} - 1.1774 \times 10^{-5}\right) \text{s}^{-1} = 5.8039 \times 10^{-5} \text{s}^{-1}$$

Thus the spin period of P at this time will be

$$\frac{2\pi}{5.8039 \times 10^{-5}} \text{ s} = 30.07 \text{ earth hours}$$

Question 5

The axes of the Hertzsprung–Russell (HR) diagram in Figure 3 represent two fundamental observable properties of stars: luminosity L (the amount of energy radiated per unit time or how bright a star is) on the y-axis and surface temperature (how hot it is on the outside, which is the only part we can directly see) on the x-axis. Typically, the luminosity is expressed in units of the Solar luminosity ($L_\odot = 3.826\times10^{26}$ W). The surface temperature of the Sun is approximately 5770 K. The most common form of the H–R diagram uses an unusual convention by representing the direction of increasing temperature values backwards compared to most graphs: temperature increases to the left and decreases to the right. Each square and pentagonal symbol in Figure 3 represents a star. The majority of stars lie on the main sequence (the cloud of points spread diagonally in the centre). You can assume that stars are in equilibrium for most of their lifetime. Note that the temperatures have been plotted on a logarithmic scale (base 10) while the logarithm (base 10) of the luminosity (in units of solar luminosity) have been used.

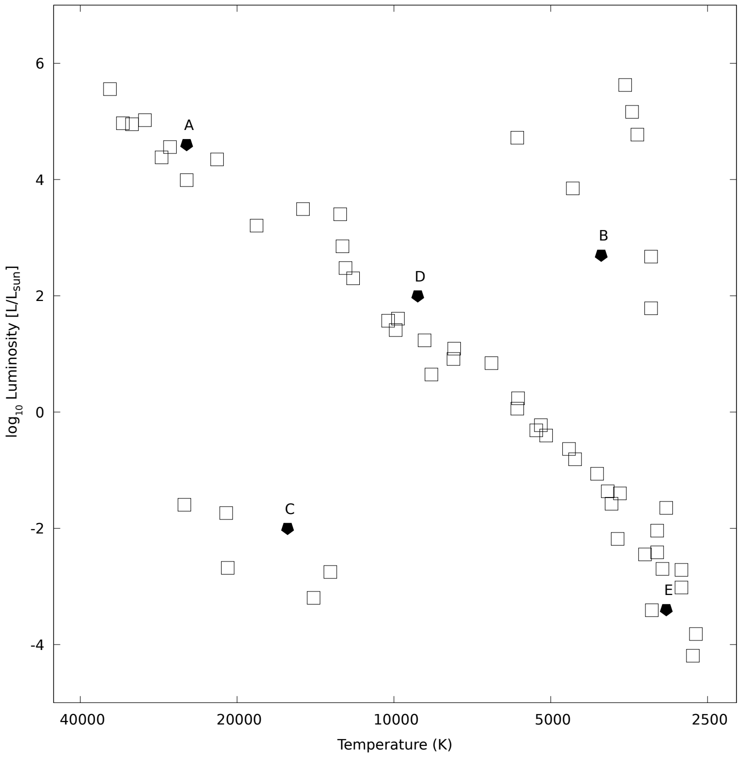


Figure 3: The HR diagram

Another property responsible for determining other characteristics of main-sequence stars is its mass M. Stars with more mass have stronger gravity, and therefore achieve higher core temperatures. Since fusion occurs only in the core, higher core temperatures produce higher fusion reaction rates, and this leads to higher luminosity. The lifetime of a star is usually defined as the amount of time that star spends fusing hydrogen into helium in its core, which is the time the star spends on the main sequence. That time is determined by the amount of hydrogen fuel initially in the star's core (which is proportional to its mass) and the rate at which the star uses that fuel. For main sequence stars $L \propto M^{3.5}$.

(a) Order the radii (increasing order) of stars labeled A, B, C, D, E (in Figure 3 and Table 1) [5]

(b) For main sequence stars, A and E, find their lifetime relative to the lifetime of the Sun. [2]

Table 1: Properties of Stars

Star Label	Temperature (K)	Luminosity ($L_{\text{star}}/L_{\odot}$)
A	25000	40000
B	4000	500
C	16000	0.01
D	9000	100
E	3000	0.0004

Solution:

(a) Given that a star is in equilibrium we can treat the emitted radiation as a blackbody. We have

$$L_{\text{star}} = 4\pi R_{\text{star}}^2 \sigma T_{\text{star}}^4$$

Writing this expression in terms of the parameters of the Sun we get

$$\frac{L_{\text{star}}}{L_{\odot}} = \left(\frac{R_{\text{star}}}{R_{\odot}}\right)^2 \left(\frac{T_{\text{star}}}{T_{\odot}}\right)^4$$

or

$$\left(\frac{R_{\text{star}}}{R_{\odot}}\right) = \left(\frac{L_{\odot}}{L_{\odot}}\right)^{1/2} \left(\frac{T_{\text{star}}}{T_{\odot}}\right)^{-2}$$

$$\log_{10}\left(\frac{R_{\text{star}}}{R_{\odot}}\right) = \frac{1}{2}\log_{10}\left(\frac{L_{\text{star}}}{L_{\odot}}\right) - 2\log_{10}\left(\frac{T_{\text{star}}}{T_{\odot}}\right)$$

We tabulate these values to compute $\log_{10}\left(\frac{R_{\text{star}}}{R_{\odot}}\right)$

Star Label	$\log_{10}(L_{\text{star}}/L_{\odot})$	$\log_{10}(T_{\text{star}}/T_{\odot})$	$\log_{10}(R_{\text{star}}/R_{\odot})$	$(L_{\text{star}}/L_{\odot})/T_{\text{star}}^4 \propto R_{\text{star}}^2$
A	4.602	0.636	1.029	1.02×10^{-13}
B	2.699	-0.159	1.668	1.95×10^{-12}
C	-2.0	0.443	-1.886	1.53×10^{-19}
D	2.0	0.193	0.614	1.53×10^{-14}
E	-3.398	-0.284	-1.131	4.94×10^{-18}

Table 2: Radius of stars

The radius of stars in increasing order is: $R_C < R_E < R_D < R_A < R_B$

(b) The amount of hydrogen (fuel) fusing into helium in a star's core is proportional to its mass. A fraction of this rate at which this fuel is used up is liberated as luminosity. The lifetime of the main sequence star $M_{\text{star}} = 10M_\odot$ is therefore:

$$t_{\text{star}} = \frac{\text{Amount of Fuel}}{\text{Rate of Fuel Used}} \propto \frac{M_{\text{star}}}{L_{\text{star}}} \propto \frac{L_{\text{star}}^{1/3.5}}{L_{\text{star}}} \propto L_{\text{star}}^{-5/7}$$

In units of solar parameters

$$\left(\frac{t_{\text{star}}}{t_\odot}\right) = \left(\frac{L_{\text{star}}}{L_\odot}\right)^{-5/7}$$

$$t_{\text{star}}^{\text{A}} \approx \frac{t_\odot}{1937} \qquad t_{\text{star}}^{\text{E}} \approx 267\ t_\odot$$

Question 6

Refer to the attached Figure . Three identical gas clouds are marked as A, B, and C. Each gas cloud emits EM radiation of luminosity L at a spectral line having the rest-frame frequency F. The gas clouds A, B, and C are in circular motion, in the counterclockwise direction, with identical speed V, in three different co-centric circles with radius R, 2R, and 3R respectively. A Telescope is situated at the point T in the circle having radius 2R, also in circular motion with speed V in the counterclockwise direction. The center of the circles is marked as O. At an instance, $\angle TAO = 90°$ and the segment CTOB forms a straight line.

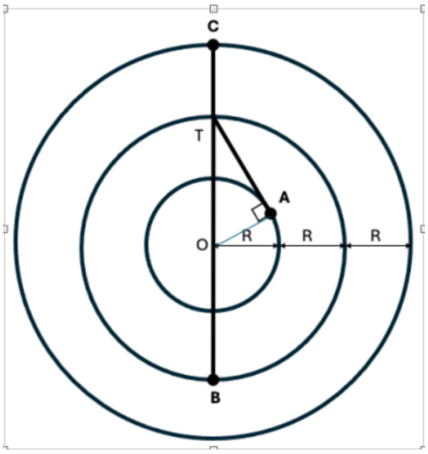


Figure 4: Locations of gas clouds

Answer the following:

1. Estimate the magnitude of line-of-sight velocity of the gas cloud A, as seen by the telescope T. [4]

2. Estimate the ratio of flux received at T from gas clouds A, B, and C. [2]

3. Which cloud(s) will show maximum Doppler frequency shift in the spectral line signal received at T. Explain your answer. [2]

Solution:

1. $TA = \sqrt{TO^2 - OA^2} = \sqrt{4R^2 - R^2} = R\sqrt{3}$

   Using the indentity in triangle TOA:

$$\frac{\sin A}{TO} = \frac{\sin T}{OA} = \frac{\sin O}{TA}$$

$$\frac{\sin 90}{2R} = \frac{\sin T}{R} = \frac{\sin O}{R\sqrt{3}}$$

Hence $\sin T = \frac{1}{2}$ and $\sin O = \frac{\sqrt{3}}{2}$; $T = 30°$ and $O = 60°$

The horizontal component of velocity of T = V, The vertical component of velocity of T = 0

For cloud A (as seen by T), the l-o-s (i.e. AT) component =

Component of A along AT + Component of T along TA

= V - V cos(60) = V - V/2 = V/2 (answer)

2. D^2 to cloud $A = (TA)^2 = 3R^2$

   D^2 to cloud $B = (TB)^2 = 16R^2$

   D^2 to cloud $C = (TC)^2 = R^2$

   Following inverse square law, flux is proportional to $1/D^2$; where D is distance

   Hence their respective flux ratios at T will be: $\frac{1}{3} : \frac{1}{16} : 1$

3. Doppler shift in frequency is directly proportional to the line-of-sight component of the relative velocity of the two moving objects.

   For cloud A (as seen by T), the l-o-s (i.e. AT) component = Component of A along AT + Component of T along TA = V - V cos(60) = $\frac{V}{2}$ (towards the T)

   For cloud B (as seen by T), the l-o-s (i.e. TB) component = Component of B along TB + Component of T along TB = 0 + 0 = 0

   For cloud C (as seen by T), the l-o-s (i.e. TC) component = Component of C along TC + Component of T along TC = 0 − 0 = 0

   Hence, only cloud A will have a non-zero, and so the maximum Doppler shift in frequency.

Question 7

A spacecraft is in a circular orbit around the Sun with an orbital radius of 1 A.U. The spacecraft deploys a panel of area A with the normal of the panel pointing towards the Sun. The panel is made of a perfectly reflecting material. What can be the maximum mass M of the spacecraft **(including that of the panel)** such that the spacecraft escapes the solar system? [5]

Solution:

Given that this is a circular orbit, we have, before unfurling of the sail:

$$Mr\dot{\phi}^2 = \frac{GM_\odot M}{r^2}$$

Here, r is the orbital radius and other symbols have their usual meaning. We assume that $M \ll M_\odot$ in the following. Thus, we have for angular momentum J:

$$\frac{J^2}{Mr^3} = Mr\dot{\phi}^2 = \frac{GM_\odot M}{r^2}$$

Hence,

$$\frac{J^2}{Mr^2} = \frac{GM_\odot M}{r} = U(r)$$

where $U(r)$ is the gravitational potential energy. In case of a circular orbit, there is no radial motion. Hence we can write for the total energy:

$$E = \frac{1}{2}M\left[\dot{r}^2 + r^2\dot{\phi}^2\right] - \frac{GM_\odot M}{r} = \frac{J^2}{2Mr^2} - \frac{GM_\odot M}{r} = -\frac{GM_\odot M}{2r}$$

When the sail is unfurled, we have to add the contribution of the radiation pressure from the Sun. The force on the object (and the sail), which is still radial, now becomes.

$$f = -\frac{GM_\odot M}{r^2} + \frac{2L_\odot A}{4\pi c r^2}$$

Thus the potential energy changes to:

$$U_{tot} = -\frac{GM_\odot M}{r} + \frac{L_\odot A}{2\pi c r}$$

If the sail is unfurled quickly then there is no change in the Kinetic energy or the gravitational potential energy. The total energy after unfurling the sail is:

$$E = -\frac{GM_\odot M}{2r} + \frac{L_\odot A}{2\pi c r}$$

The object will shift to an escape orbit if the Energy is positive. Thus, we have:

$$\frac{L_\odot A}{2\pi c r} > \frac{GM_\odot M}{2r}\ ,$$

implying:

$$\frac{A}{M} > \frac{\pi c G M_\odot}{L_\odot} = 329 \text{ sq.m/kg}$$

Therefore we have:

$$M < \frac{A}{329}\ \text{kg}$$

The same expression can be derived if one uses the work energy theorem to derive the form of the potential energy and then require $E > 0$.

Another approach, where we require the net force to be away from the Sun gives an answer that differs by a factor of two as it does not take the initial conditions into account.

Question 8

Let us consider a distant point-like object, say a star, which is emitting radiation isotropically. The star is surrounded by numerous hydrogen clouds, uniformly distributed over a large radius r. The radiation from the star after getting scattered from the gas clouds arrive as parallel rays to a distant observer who collects these with a lens, as depicted in Figure 5. Derive the expression for the locus of clouds that will result in a fixed time delay of $\tau = R/c$ (known as an isodelay surface) to echo the brightness change of the star? For simplicity, consider a cloud distribution in 2-dimensions with the star at $(x, y) = (0, 0)$. [5]

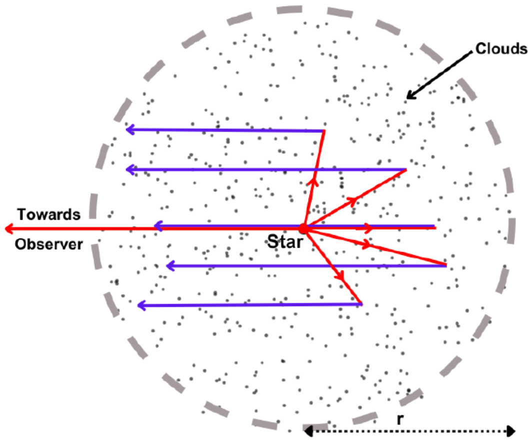


Figure 5: Star is in the centre and all the dots are the clouds absorbing radiation incident on them from the star and then emitted as line radiation towards a distant observer.

Solution:

For a fixed time delay of $\tau = R/c$ the photons emitted after absorption as line emission, needs to travel an extra path of R distance in comparison to the photon directly received from the star. This is equivalent to the situation where all these emitted photons start their journey from a perpendicular line behind the star at a distance R, as depicted by the verticle dotted line in Figure 6. Therefore the locus of clouds will be those clouds whose distance from the star (shown by the solid line with arrow) is equal to their perpendicular distances from this line (shown by a dotted line).

Let the points on the locus (in 2-dimension) be $P(x, y)$ and the star be at $(x, y) = (0, 0)$ and the line is $x = R$, then,

(i) The distance of $P(x, y)$ from the point $(0, 0)$:

$$\sqrt{x^2 + y^2}$$

(ii) The perpendicular distance from the line $x = R$:

$$|x - R|.$$

Condition of equal distances imply

$$\sqrt{x^2 + y^2} = |x - R|$$

Simplify:

$$y^2 = -2Rx + R^2$$

Rewriting:

$$y^2 = -2R\left(x - \frac{R}{2}\right)$$

which is a parabola with focus at (0,0), vertex at $(R/2{,}0)$, and Directrix as $x = R$ as depicted in Figure 6.

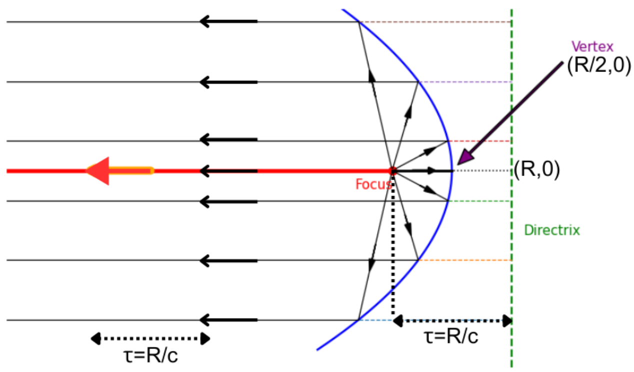


Figure 6: The parabola shown in the figure is the locus of points that are equidistant from the Directrix and the focus. Here for illustration, we kept focus at (0,0), vertex at $(R/2{,}0)$ and Directrix as $x = R$.

Question 9

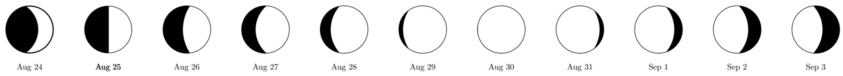
On August 16, 2023, it was new Moon and on August 23, 2023, 6 PM IST (the Chandrayan-3 landing event) it was sixth day of the waxing moon. Nearly 30% of Moon was illuminated as illustrated below:

The orbital plane of Moon is tilted by about 5°, with respect to the ecliptic plane, and it orbits in an elliptical orbit. However, assume that the path of Moon around the Earth is coplanar with ecliptic and the orbit is circular.

An observer from Moon (nearside close to equator) will see the illuminated portion of the Earth change from August 23, 2023 onward as illustrated in ______ (fill (a), (b), (c) or (d)) and explain your choice (Ignore the difference in the size, if any.). [5]

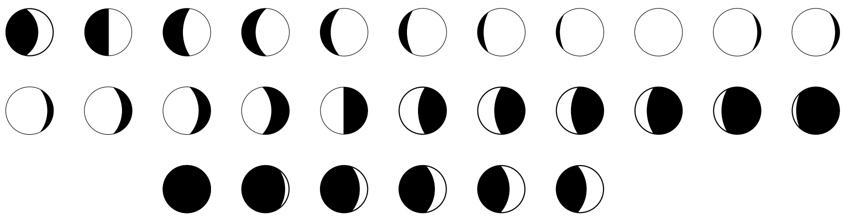
(a)

(b)



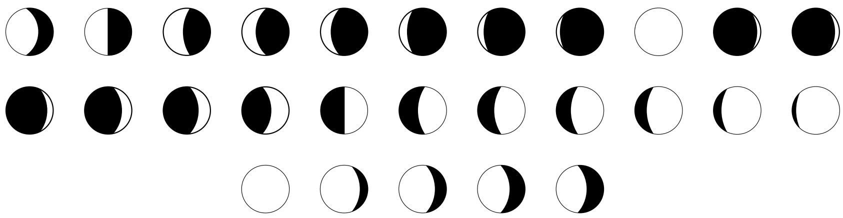
(c)

Every 24 hours earth will undergo a cycle of full phases, starting from waxing cresent, at the time of landing.



(d)

Every 24 hours earth will undergo a cycle of full phases, starting from waning cresent, at the time of landing.



Solution:

Option (a) is correct. The illumination of Earth as seen from nearside of Moon will be opposite to that of the Moon from Earth. August 24, 2024 will see Earth 40% dark and 100% dark on August 30, 2024 as it will be full Moon from Earth. On September 3, 2023 the Earth will be approximately 40% illuminated.

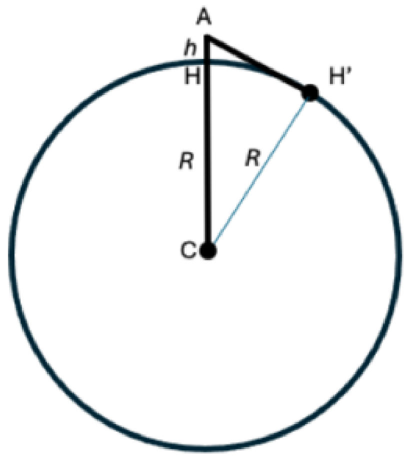
Note: In case the student has mentioned that the black portion is illuminated then (b) is also the correct answer. However by default the white portion is considered illuminated and (a) is correct.

Question 10

Consider Earth as a sphere of radius 6400 km. At the time of equinox (i.e. center of the Sun's disk is at declination 0.0 deg), an observer is standing exactly at North pole with a drone (a flying device) camera. The observer wishes to see the full disk of the Sun, which subtends an angle of 0.009 radian to the observer. At what minimum height, flying above the North pole, the drone camera can see the full disk of the Sun. Assume that the local horizon is clear without any geographical obstructions (land, trees, building etc.) to the horizon and sun-rays are parallel. Also ignore all light-ray bending effects such as atmospheric refraction etc. [5]

Solution:

Refer to the Figure below that depicts the geometry in the question :



$AH = h$ where $h << R$ (Radius of Earth)

By geometry, $\angle AH'C = 90°$

$\angle HCH'$ should be $0.009/2$ radians $= 0.0045$ radians to see the full disk of the Sun at height h.

(Note that one half of the Sun's disk is already visible at North pole on the equinox day at $h = 0$.)

Horizon distance AH' at height h can be estimated as follows –

$AH' = \sqrt{(R+h)^2 - R^2} = \sqrt{h^2 + 2Rh}$

Since $h << R$; $AH' = \sqrt{2Rh}$

$\sin(\angle HCH') = 0.0045 = \frac{AH'}{AC} = \frac{\sqrt{2Rh}}{R+h} = \sqrt{\frac{2h}{R}}$

(Note that angle HCH' is small, hence $\sin(\angle HCH') = \angle HCH'$ in radians)

$$0.0045 = \sqrt{\frac{2h}{6400}}$$

Solving the above: $h = 3200 \times 0.0045 \times 0.0045 = 0.065$ km or 65 meters.

Answers without using the small angle approximation are also accepted.

Question 11

A powerful luminous source is orbiting almost touching to the surface (at equator) of a spherical white dwarf in a circular orbit. The rotation orbit is perpendicular to the line joining star and us. Variation of light received from this source has A periodicity of 8 seconds. What is the density of white dwarf? [5]

Two orientations are possible [**1**]

- One where the orbital plane is perpendicular to our line of sight so that source is always visible (light source at the edge of the projected size of the White dwarf). In this case no variation is expected.
- The other is when our line of sight lies in the plane of rotation (White dwarf shadowing the light source) and since variation and periodicity is observed/reported, the is the desired configuration

A maximum possible spin is such that centripetal and gravity equals at its equator

$$\frac{v^2}{R} = \frac{GM}{R^2} \Longrightarrow v = \sqrt{\frac{GM}{R}}$$

[**1**]

Time period of rotation is

$$T = \frac{2\pi R}{v} = \frac{2\pi R^{3/2}}{\sqrt{GM}}$$

[**1**]

and

$$M = \frac{4}{3}\pi R^3 \rho \Longrightarrow T = \frac{2\pi R^{3/2}}{\sqrt{G4/3\pi R^3 \rho}} = \sqrt{\frac{3\pi}{G}}\rho^{-1/2}$$

[**1**]

For a sighting period of 8 seconds

$$\rho = \frac{3\pi}{GT^2} = (2.20 \pm 0.01) \times 10^9 \ kg/m^3$$

[**1**]

Question 12

A photon leaving the surface of non-spinning compact stars like white dwarf or Neutron star will face intense gravity, leading to a change in the wave energy E associated with the photon (this phenomenon is generally called gravitational redshift). For gravity effect on photons, an effective mass m can be assumed through the relation $E = mc^2$. Estimate the fractional change in the frequency of a photon leaving the surface of a white dwarf, Sirius B of mass M=1$M_\odot$ and radius $R = 0.008R_\odot$, when observed by a distant observer (assuming Newton's gravity). [5]

Gravitational redshift depends on mass to radius ratio. Larger the ratio, larger the effect.

For a photon leaving the surface and reaching an observer at infinity, photon total energy (Kinetic: T + Potential: U) should be conserved.

$$T_i + U_i = T_f + U_f; \Longrightarrow T_f - T_i = U_i - U_f$$

[**1**]

Since photon travels at the speed of light (a constant), change of energy will be reflected in its change of frequency ($h\nu = mc^2 \rightarrow h\Delta\nu = c^2\Delta m$; or equivalent form) [**1**]

$$U_f = 0, T_f = h\nu_f, U_i = -\frac{GMm}{R} = -\frac{GM}{R}.\frac{h\nu}{c^2}$$

$$\Delta T = T_f - T_i = h\Delta\nu$$

$$h\Delta\nu = -\frac{GM}{R}.\frac{h\nu}{c^2} \implies \frac{\Delta\nu}{\nu} = -\frac{GM}{Rc^2}$$

[**1**]

For Sirius B, M = 2×10^{30} kg, R = $0.008 \times 6.95 \times 10^8$ m, c = 3×10^8 m/s

$$\frac{\Delta\nu}{\nu} = 2.65 \times 10^{-4}$$

[**2**]

Calculus-based approach:

Since effective change in energy (mass) is basically due to the effect of gravity-potential

$$d(mc^2) = \frac{GMm}{r^2}dr$$

$$c^2 \int_{m_i}^{m_f} \frac{dm}{m} = \int_R^\infty \frac{GM}{r^2}dr$$

[**1**]

$$ln\frac{m_f}{m_i} = -\frac{GM}{c^2R}$$

$$m_f = m_i e^{-\frac{GM}{c^2R}}$$

$$m_i + \Delta m = m_i e^{-\frac{GM}{c^2 R}}$$

$$\frac{\Delta m}{m_i} = -(1 - e^{-\frac{GM}{c^2 R}})$$

[**1**]

Now, $E \equiv h\nu = mc^2 \rightarrow h\Delta\nu = c^2 \Delta m$ [**1**]
therefore,

$$\frac{\Delta \nu}{\nu} = -(1 - e^{-\frac{GM}{c^2 R}})$$

For Sirius B, $GM/Rc^2 = 2.65 \times 10^{-4} << 1$;
so,

$$\frac{\Delta \nu}{\nu} = -\frac{GM}{c^2 R} = -2.65 \times 10^{-4}$$

[**2**]