Indian National Physics Olympiad (INPhO)-2024

HOMI BHABHA CENTRE FOR SCIENCE EDUCATION
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Question Paper

Date: 04 February 2024
Time: **09:00-12:00 (3 hours)** Maximum Marks: **80**

Instructions **Roll No.:**

1. This booklet consists of 23 pages and total of 6 questions. Write roll number at the top wherever asked.

2. Booklet to write the answers is provided separately. Instructions to write the answers are on the Answer Booklet.

3. Non-programmable scientific calculators are allowed. Mobile phones **cannot** be used as calculators.

4. **Please submit the Answer Sheet at the end of the examination.** You may retain the Question Paper.

Table of Constants

Universal constant of Gravitation	G	6.67×10^{-11} N·m^2·kg^{-2}
Value of $1/4\pi\epsilon_0$		9.00×10^9 N·m^2·C^{-2}
Avogadro's number	N_A	6.022×10^{23} mol^{-1}
Acceleration due to gravity	g	9.81 m·s^{-2}
Universal Gas Constant	R	8.31 J· K^{-1}·mol^{-1}
	R	0.0821 l·atm·mol^{-1}·K^{-1}
Molar mass of Helium		4.003 g· mol^{-1}
Boltzmann constant	k_B	1.3806×10^{-23} J·K^{-1}
Earth's radius	R_E	6371km
Earth's mass	M_E	5.97×10^{24}kg
Moon's mass	M_M	7.35×10^{22}kg
Moon's radius	R_M	1737km

Question Number	**1**	**2**	**3**	**4**	**5**	**6**	**Total**
Maximum Marks	8	18	14	11	18	11	80

1. **[8 marks] An electrifying experiment**
Professor Coulomb was investigating how the magnitude of the force ($|\vec{F}|$) between two charged spheres depends on the distance between their centres. He conducted four separate experiments by placing two identical conducting spheres 1 and 2, each of radius a, at different distances d from each other. The experiments are outlined in the table below. Here Q_1 and Q_2 are the charges on the spheres 1 and 2, respectively. The measurement results are presented in the graph.

Experiment no.	a (m)	Q_1	Q_2
1	0.10	$+Q$	$+Q$
2	0.10	$+Q$	$-Q$
3	0.05	$+Q$	$+Q$
4	0.05	$+Q$	$-Q$

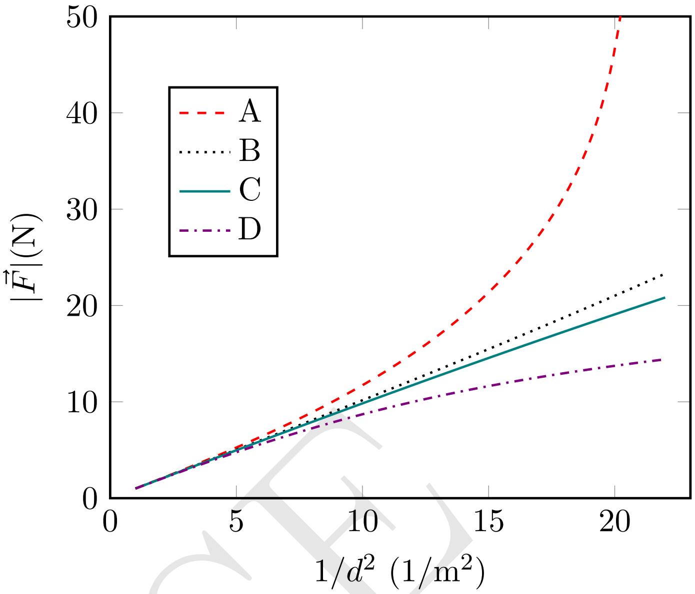
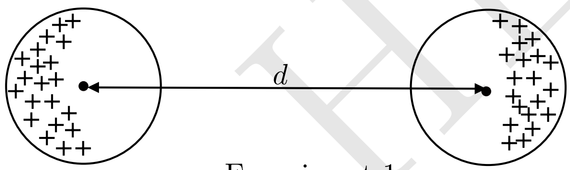
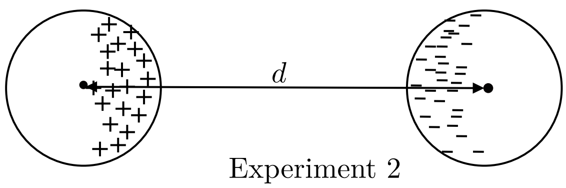


Figure out which measurement (A, B, C, D) belongs to which experiment (1, 2, 3, 4). Explain your answers in the detailed answersheet. You may draw diagrams, if necessary.

Solution: Here conducting spheres exhibit polarization when brought close, impacting the effective distance between charges. For two positively charged spheres at a distance d from their centers, polarization increases the effective distance, reducing the force. Similarly, for a positively and a negatively charged sphere, also at distance d from their centers, polarization decreases the effective distance and increases the force. The order of effective distances in the four experiments is $d_1^{\text{eff}} > d_3^{\text{eff}} > d_4^{\text{eff}} > d_2^{\text{eff}}$. This is depicted in the figure below.

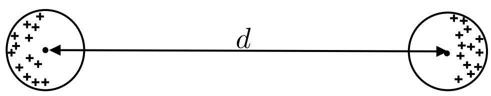


Experiment 1

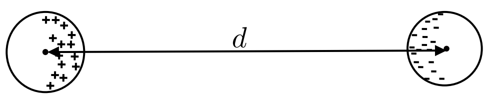


Experiment 2

The correct choices will be:
Experiment 1: D
Experiment 2: A
Experiment 3: C
Experiment 4: B



Experiment 3

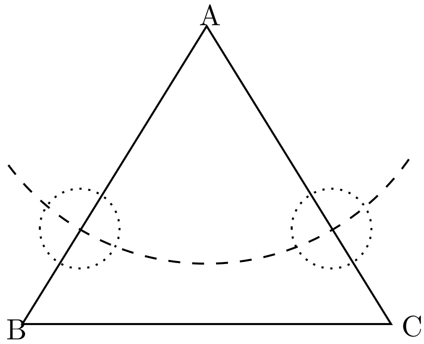


Experiment 4

2. **A Potpourri of Prism Problems**

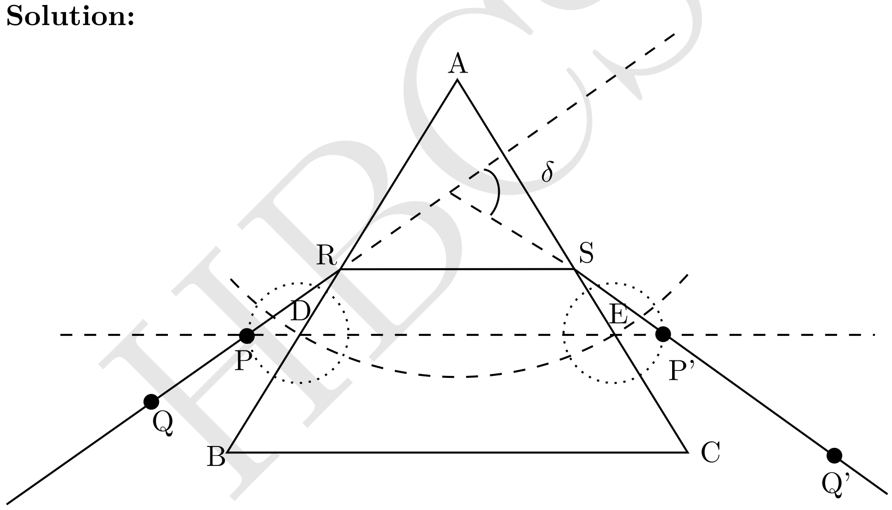
(a) **[7 marks]** In the most common method to determine the angle of minimum deviation by a prism, we record the angles of deviation (δ) for various angles of incidence (i) and then plot a graph. However, Professor Joseph proposed an ingenious idea to determine the angle of minimum deviation with just a single angle of incidence. Eager to share his breakthrough, he penned a letter to his friend, Professor Amal Nathan, outlining his method for an equilateral prism. According to Professor Joseph, the only tools required were four pins, a board, a marker pen or pencil, a scale, a protractor, and, of course, the prism. He even claimed that one may not need all the materials listed.

In his letter, Professor Joseph began to sketch a figure to illustrate the method. Here triangle ABC is the trace of the prism. The dashed curve is an arc of a circle centered at A. The dotted circles are centered on the points of intersection of the arc with the sides of the triangle. Unfortunately, he forgot to complete the figure. Can you describe the experimental method to determine the angle of minimum deviation using the unfinished figure of Professor Joseph? You must provide the following:



1. A complete ray diagram using the given figure. You may use the one already provided on the answersheet or draw a fresh one.
2. Outline of the essential experimental steps using some or all of the equipment mentioned above and nothing more. Use the detailed answer sheet for this.

Solution:

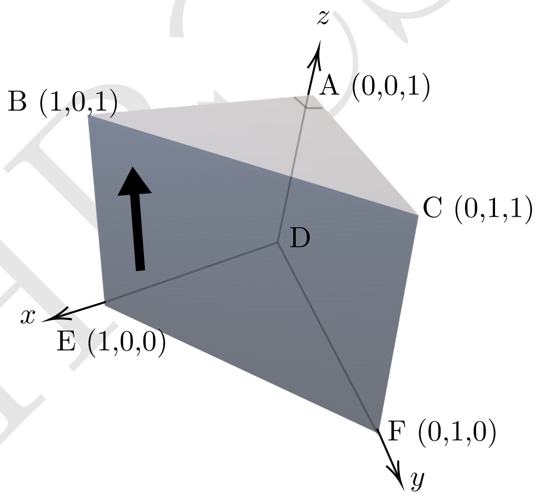


The big arc is drawn to draw a line parallel to the base BC, with A as the center. Smaller dotted circles are then drawn, centering on points D and E. The goal is to minimize the angle of deviation, which occurs when the refracted ray is parallel to the base. To observe and measure this deviation, the following steps are taken:

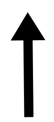
1. Draw a line passing through D and E. This is the dashed line parallel to the base BC.
2. Mark point P, where the circle centered at D intersects the line DE. Similarly, mark P'.
3. Mount pins on the points P and P'.
4. Mount 3rd pin at a position Q on the left side, such that pins at P, P', and Q appear collinear when observed from the right side refracting face of the prism.
5. In a similar manner looking from the left side, align the 4th pin labeled as Q' on the right side, such that pins at P, P', Q, and Q' appear collinear.

6. Observe the incident, refracted, and emergent rays formed by the pin's positions and the prism.
7. Extend the lines drawn to measure the angle of deviation.
8. Since PD=P'E, the refracted ray is parallel to the base, and the angle δ represents the minimum deviation.
9. Utilize a protractor to measure the angle.

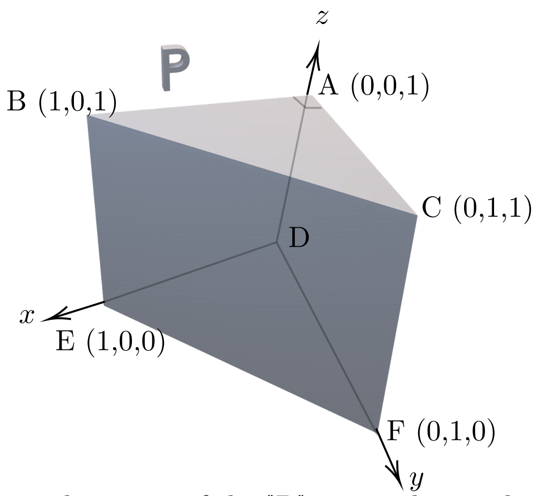
(b) **[1.5 marks]** Consider a right-angled isosceles prism as depicted below. The prism is placed on a table (x-y plane). The triangular faces are non-refracting surfaces. The refractive index of the prism is 1.50. The sides AB = AC = AD = 1 unit. The prism is positioned such that point D is at the origin, with the axes defined in the figure. An arrow-shaped object is pasted on the face BCFE of the prism as shown. Draw the image of the object as seen by an observer in front of the face BCFE.

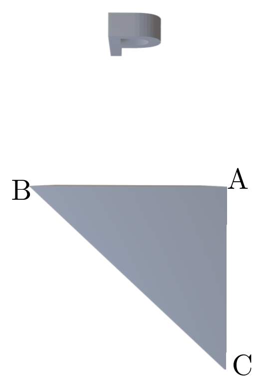


Solution: The formed image is virtual and located behind face ACDF when seen from face BCFE. The pasted arrow is symmetric about edge AD. It undergoes two total internal reflections, resulting in the image being identical to the object but laterally shifted.



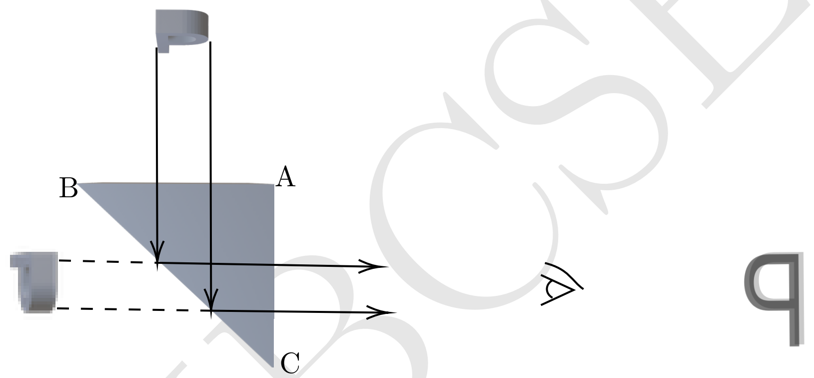
(c) **[1.5 marks]** Now an object, shaped like the letter "P" as illustrated in the left figure below, is held in front of the face ABED of the prism, placed on a table (x-y plane). The corresponding top view of this configuration is also presented in the right figure below.



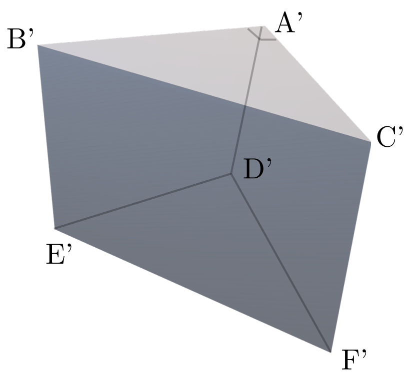


Draw the image of the "P" as seen by an observer in front the face ACFD. Additionally, draw a qualitative ray diagram illustrating the image formation.

Solution: The formed image is virtual and positioned behind face BCFE when observed facing face ACDE. It undergoes total internal reflection. Ray diagram and the resultant image is shown below.



(d) **[5 marks]** In this part, alongside the setup of part (c) with the prism ABCDEF and the object "P" positioned in front of its face ABED, we introduce another identical prism, A'B'C'D'E'F' (see below).



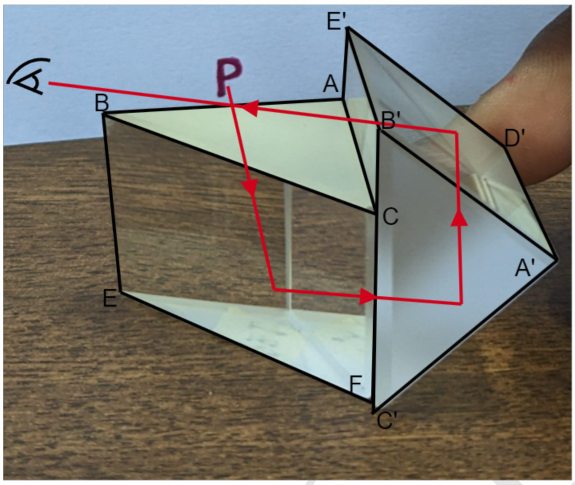
A specific experiment requires placing the prism A'B'C'D'E'F' in combination with the existing setup so that the following image (as shown below) of the object "P" can be obtained.

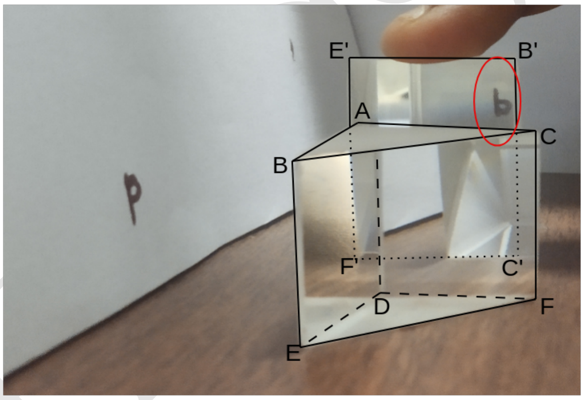
Without disturbing the prism ABCDEF, where and how would you hold the prism A'B'C'D'E'F' to achieve this resultant image?

Provide your answer in terms of the coordinates of the vertices of the prism A'B'C'D'E'F', taking vertex D of the first prism as the origin. Additionally, specify that to view such an image, the viewer should be facing which particular face of the prism.

Hint: The position of the second prism is such that any one of the parts (such as at least one of the vertices, edges, or faces) touches the table. The viewer should be positioned in a way that allows a clear line of sight to the area where the image is formed, ensuring an unobstructed view of the desired image.

Solution:





The light rays from object "P" initially pass through faces ABED of prism 1 at a normal angle. Subsequently, due to total internal reflection, the rays are reflected from face BCFE. The reflected light exits prism 1 through face ACFD and enters prism 2 through face E'B'C'F'. Inside prism 2, the light undergoes total internal reflection twice, occurring at faces A'C'F'D' and A'D'E'B'. The image formed is virtual and located behind the face C'F'E'B' face of the prism.

Face **C'F'E'B'** of the prism 2 can be kept in the plane **YZ**, touching the face **CADF** of the prism 1.

The position of the observer is in the plane **YZ and facing C'F'E'B' face of the second prism.**

Coordinates of the face **C'F'E'B'** are

C': $(0, 1, 0)$

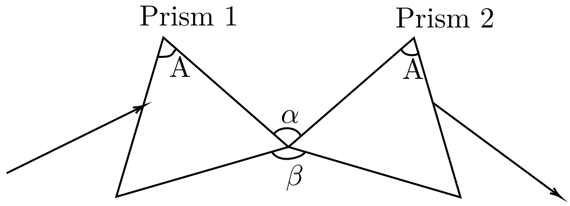
F': $(0, 0, 0)$

E': $(0, 0, \sqrt{2})$

B': $(0, 1, \sqrt{2})$

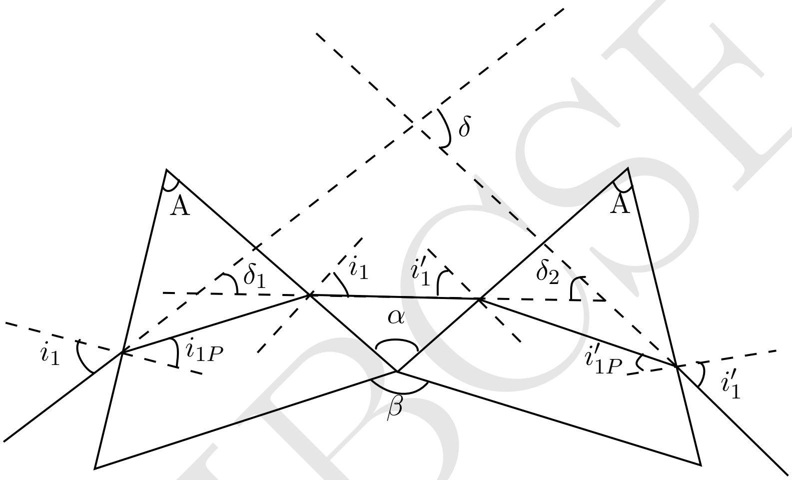
Alternate configuration, leading to the correct image are equally credited.

(e) **[3 marks]** In a spectrograph, two equilateral prisms denoted as 1 and 2 with refractive indices $\mu_1 = 1.50$ and $\mu_2 = 1.68$, respectively, are placed one after another (see the figure below). The incident ray is shown on the left and the final emergent ray is shown on the right.



Find the angle (β) between the bases of the two prisms if each prism is individually adjusted for minimum deviation for the respective incident rays. Obtain the total deviation δ of the beam of light in this configuration. Express your answers in terms of μ_1, μ_2, A. Calculate β and δ in degrees.

Solution: Let angle of incidence be i_1 and for this ray, the angle of emergence from the second prism be i_1'. The angle of refraction at first prism be i_{1P}. The angle of incidence of light for second prism is i_1', as it is adjusted for minimum deviation. Angle of refraction for ray going from air to 2nd prism be i_{1P}'.



$$\sin i_1 = \mu_1 \sin \frac{A}{2} \tag{2.1}$$
$$\sin i_1' = \mu_2 \sin \frac{A}{2} \tag{2.2}$$

From geometry

$$\alpha = i_1 + i_1' \tag{2.3}$$
$$\alpha = \sin^{-1}\left(\mu_1 \sin \frac{A}{2}\right) + \sin^{-1}\left(\mu_2 \sin \frac{A}{2}\right) \tag{2.4}$$
$$\beta = 240 - (i_1 + i_1') \tag{2.5}$$
$$\beta = 240 - (\sin^{-1}\left(\mu_1 \sin \frac{A}{2}\right) + \sin^{-1}\left(\mu_2 \sin \frac{A}{2}\right)) \tag{2.6}$$

For $\mu_1 = 1.50$, $\mu_2 = 1.68$ and $A = 60°$, we get

$$\alpha = 105°44' \tag{2.7}$$
$$\beta = 134°16' \tag{2.8}$$

From figure and geometry If the angle of minimum deviation of the first prism is δ_1 and the second prism is δ_2., then

$$A + \delta_1 = 2i_1 \tag{2.9}$$
$$A + \delta_2 = 2i_1' \tag{2.10}$$

The total deviation produced by two prisms is

$$\delta = \delta_1 + \delta_2 \tag{2.11}$$
$$\delta = 2(i_1 + i_1' - A) \tag{2.12}$$
$$\delta = 2(\sin^{-1}\left(\mu_1 \sin\frac{A}{2}\right) + \sin^{-1}\left(\mu_2 \sin\frac{A}{2}\right) - A) \tag{2.13}$$
$$= 91°28' \tag{2.14}$$

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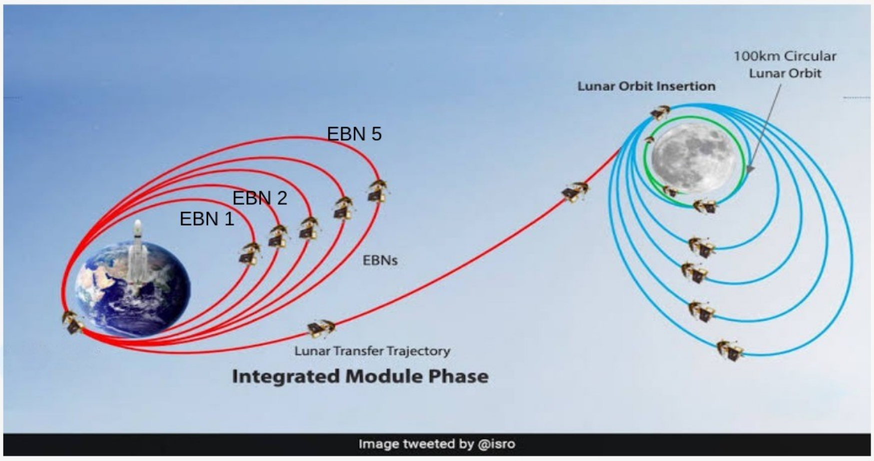
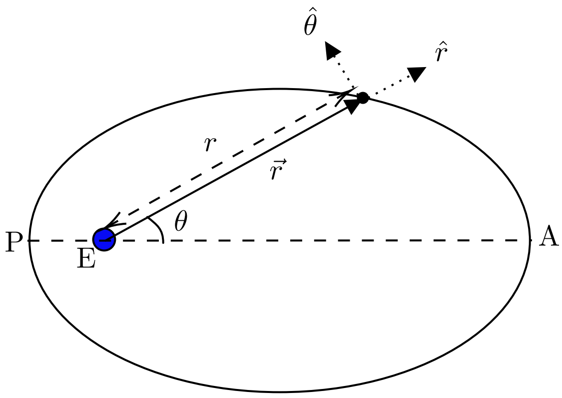
3. **Chandrayaan-3**

On July 14, 2023, India's lunar mission satellite, *Chandrayaan-3*, was successfully launched by the Indian Space Research Organization (ISRO). *Chandrayaan-3* (mass m =3900 kg) was taken to the Moon through a series of Earth Bound Manoeuvres (elliptical) orbits (EBNs) as depicted in the figure below. In this problem, we will explore the physics governing some part of its journey, employing a simplified model. For all parts of this problem except part (f), we consider *Chandrayaan-3* to be moving only under the influence of Earth's gravity (a central force).



(a) **[6 marks]** Upon launch, *Chandrayaan-3* entered an elliptical orbit around Earth, with Earth at one of the foci (E) as shown below. The points P and A are the perigee (nearest point from the Earth) and apogee (farthest point from the Earth), respectively. We introduce the polar coordinate system (r,θ), where $\vec{r}$ is the vector from the centre of the Earth (origin) to the satellite, and θ is the angle that $\vec{r}$ makes with the major axis (PA= $2a$). The directions of unit vectors $\hat{r}$ and $\hat{\theta}$ are shown in the figure.



The equation of the ellipse can be written in polar coordinates as

$$r = \frac{r_0}{(1 - e\cos\theta)}$$

where e is eccentricity of the orbit $(0 < e < 1)$ and r_0 is called the latus rectum.
The velocity $\vec{v}$ of the satellite in polar coordinates can be written as

$$\vec{v} = v_r\hat{r} + v_t\hat{\theta} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

where $v_r = \dot{r}$ is the "radial" speed and $v_t = r\dot{\theta}$ is the "tangential" speed.
Make schematic plots of the speeds v_r and v_t as functions of θ over one full orbit. Mark any significant points in the plots in terms of a, e, and other variables.

Solution: It is given that

$$r = \frac{r_0}{(1 - e\cos\theta)} \tag{3.1}$$

$$\therefore v_r = \dot{r} = -\frac{r_0 e \sin\theta\dot{\theta}}{(1 - e\cos\theta)^2} \tag{3.2}$$

Since the force is central, angular momentum l is conserved. The conserved angular momentum is given by

$$l = mr^2\dot{\theta} \tag{3.3}$$

$$\implies \dot{\theta} = \frac{l}{mr^2} \tag{3.4}$$

Substituting above equation and expression of r in v_r, after simplifying, we get

$$v_r = -\frac{el\sin\theta}{mr_0} \tag{3.5}$$

We know that

$$v_t = \frac{l}{mr} \tag{3.6}$$

Substituting the value of r, we get

$$v_t = \frac{l(1 - e\cos\theta)}{mr_0} \tag{3.7}$$

From above equation, the value of v_t is maximum, when $\cos\theta$ is minimum. i.e. $\theta = -\pi$. The maximum value of v_t

$$v_t^{\max} = \frac{l(1+e)}{mr_0} \tag{3.8}$$

From Eq. (3.7), v_t is minimum, when $\cos\theta$ is maximum. i.e $\theta = 0$. The minimum value of v_t

$$v_t^{\min} = \frac{l(1-e)}{mr_0} \tag{3.9}$$

Since the velocities at P and A are purely tangential, and P is closer to earth as compared to A, hence the tangential speed is maximum at P. Let the velocity of the satellite in the given orbit at P be v_p and velocity at A be v_a. Conservation of energy gives:

$$\frac{1}{2}mv_a^2 - \frac{GMm}{r_a} = \frac{1}{2}mv_p^2 - \frac{GMm}{r_p} \tag{3.10}$$

Conservation of angular momentum gives

$$mv_a r_a = mv_p r_p \tag{3.11}$$

$$v_p = v_a\frac{r_a}{r_p} \tag{3.12}$$

Substituting the above equation into the energy conservation equation, we get

$$v_p = \sqrt{\frac{GM}{a}\frac{(1+e)}{(1-e)}} \tag{3.13}$$

Similarly

$$v_a = \sqrt{\frac{GM}{a}\frac{(1-e)}{(1+e)}} \tag{3.14}$$

Hence the conserved angular momentum can be written as

$$l = mv_a r_a = mv_a r_a \tag{3.15}$$

On simplification, we get

$$l = mr_o\sqrt{\frac{GM}{a(1-e^2)}} \tag{3.16}$$

v_r and v_t can also be written as

$$v_r = -e\sqrt{\frac{GM}{a(1-e^2)}}\sin\theta \tag{3.17}$$

$$v_t = \sqrt{\frac{GM}{a(1-e^2)}}(1-e\cos\theta) \tag{3.18}$$

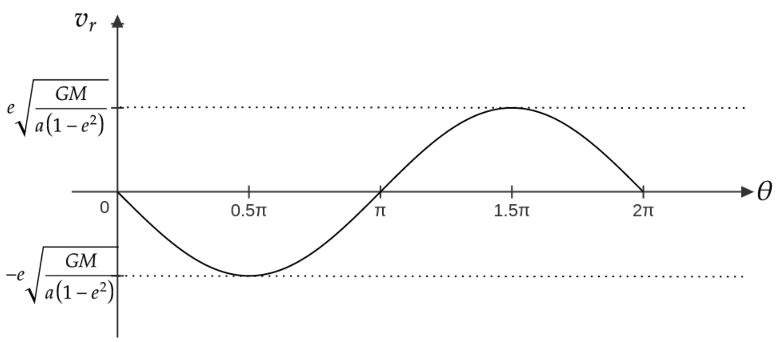
Similarly v_r^{max}, v_r^{min}, v_t^{max} and v_t^{min} can also be written as

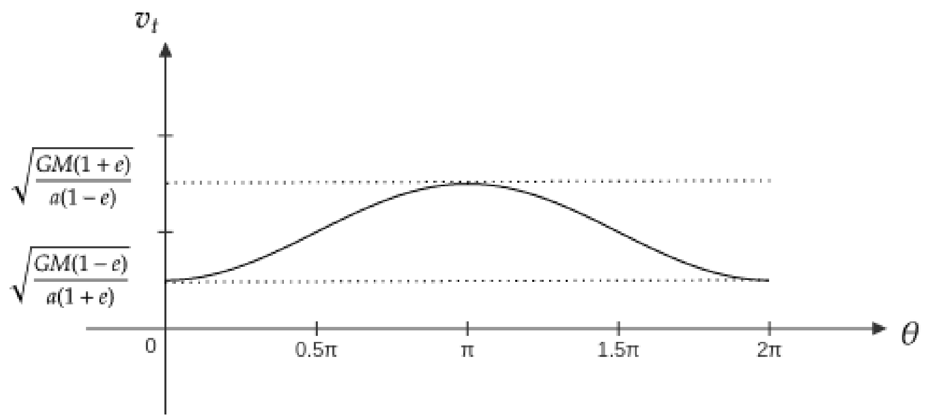
$$v_r^{max} = e\sqrt{\frac{GM}{a(1-e^2)}} \quad \text{at } \theta = 3\pi/2 \tag{3.19}$$

$$v_r^{min} = -e\sqrt{\frac{GM}{a(1-e^2)}} \quad \text{at } \theta = \pi/2 \tag{3.20}$$

$$v_t^{max} = \sqrt{\frac{GM}{a(1-e^2)}}(1+e) = \sqrt{\frac{GM(1+e)}{a(1-e)}} \quad \text{at } \theta = \pi \tag{3.21}$$

$$v_t^{min} = \sqrt{\frac{GM}{a(1-e^2)}}(1-e) = \sqrt{\frac{GM(1-e)}{a(1+e)}} \quad \text{at } \theta = 0 \text{ and } 2\pi \tag{3.22}$$





(b) **[1.5 marks]** Obtain an expression for the total energy (E) of the orbiting satellite in terms of a and other constants.

Solution: Let

$$\frac{v_a}{r_p} = \frac{v_p}{r_a} = C \tag{3.23}$$

Putting Eq. (3.23) in Eq. (3.10), we get

$$\frac{1}{2}mC^2r_p^2 - \frac{GMm}{r_a} = \frac{1}{2}mC^2r_a^2 - \frac{GMm}{r_p} \tag{3.24}$$

solving above equation, we get

$$C^2 = \frac{2GM}{(r_p + r_a)r_ar_p} \tag{3.25}$$

Again using Eq. (3.23) and value of C^2 from above equation in LHS of the Eq. (3.24), we get

$$E = \frac{1}{2}m\frac{2GMr_p^2}{r_ar_p(r_a + r_a)} - \frac{GMm}{r_a} \tag{3.26}$$

Solving above equation, and substituting $r_a + r_p = 2a$, we get

$$E = -\frac{GMm}{2a} \tag{3.27}$$

(c) **[1 marks]** Plot the kinetic energy (KE) of the satellite as a function of θ over one full orbit. Mark any significant points in terms of a, e, and other variables.

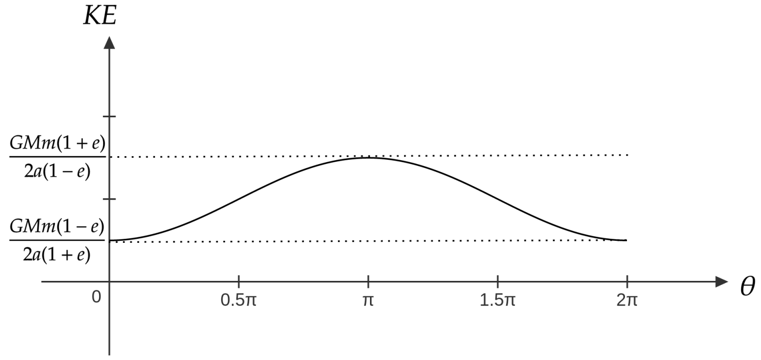
Solution:

$$KE = \frac{1}{2}mv^2 \tag{3.28}$$

where $v^2 = v_r^2 + v_t^2$. Substituting the value of v_r and v_t in the above equation, we get

$$KE = \frac{GMm(1 + e^2 - 2e\cos\theta)}{2a(1 - e^2)} \tag{3.29}$$

For $\theta = 0$, $KE = KE_{min} = \frac{GMm(1-e)}{2a(1+e)}$, and for $\theta = \pi$ $KE = KE_{max} = \frac{GMm(1+e)}{2a(1-e)}$



(d) **[1.5 marks]** The perigee and apogee of the elliptical orbit in part (a) are 200 km and 36500 km, respectively. It is generally described as a (200×36500) km orbit. Here the distances are defined from the surface of the Earth. Calculate the period of rotation T (in hr) of *Chandrayaan-3* in this orbit.

Solution: The period of the orbit is given by

$$T^2 = \frac{4\pi^2}{GM}a^3 \tag{3.30}$$

where $a = \frac{r_p + r_a}{2}$, where r_p is perigee distance given by $r_p = 200\text{km} + R_E$ and r_a is apogee distance given by $r_a = 36500\text{km} + R_E$ and i.e. $a = 6371 + \frac{(200+36500)\text{km}}{2} = 24721\text{km}$

$$T = \sqrt{\frac{4 \times \pi^2}{6.67 \times 10^{-11} \times 5.972 \times 10^{24}} 24721^3 \times 10^9} \tag{3.31}$$

$$= 10.75 \text{ hr} \tag{3.32}$$

(e) **[2.5 marks]** To move *Chandrayaan-3* from the first orbit (in part (d)) to another elliptical orbit EBN-1, an instantaneous boost was applied at perigee by changing the velocity by Δv, without altering the direction. This changed the apogee to 41800 km above Earth's surface while keeping the perigee unchanged. Calculate Δv.

Solution: since the velocities at P and A are purely tangential, and P is closer to earth as compared to A, hence the tangential speed is maximum at P. Let the velocity of the satellite in the given orbit at P be v_p and velocity at A be v_a. Conservation of energy gives:

$$\frac{1}{2}mv_a^2 - \frac{GMm}{r_a} = \frac{1}{2}mv_p^2 - \frac{GMm}{r_p} \tag{3.33}$$

Conservation of angular momentum gives

$$mv_a r_a = mv_p r_p \tag{3.34}$$

$$v_p = v_a \frac{r_a}{r_p} \tag{3.35}$$

Substituting the above equation into the energy conservation equation, we get

$$v_p = \sqrt{\frac{2GMr_a}{r_p(r_p + r_a)}} \tag{3.36}$$

$$v_p = \sqrt{\frac{GMr_a}{ar_p}} \tag{3.37}$$

$$v_p = \sqrt{\frac{GM}{a}\frac{(1+e)}{(1-e)}} \tag{3.38}$$

Similarly

$$v_a = \sqrt{\frac{2GMr_p}{r_a(r_p + r_a)}} \tag{3.39}$$

$$v_a = \sqrt{\frac{GMr_p}{ar_a}} \tag{3.40}$$

$$v_a = \sqrt{\frac{GM}{a}\frac{(1-e)}{(1+e)}} \tag{3.41}$$

From Eq. (3.36)

$$v_p = \sqrt{\frac{2GMr_a}{r_p(r_p + r_a)}} \tag{3.42}$$

$$v_p = 10.251 \text{ km/s} \tag{3.43}$$

In the new orbit, the perigee distance is kept the same, and the apogee distance changed to 41800 km, hence $r_a = 41800 + R_E = 48171$km and r_p remains as it is. Hence, the new velocity at the perigee distance is v'_p Using the expression Eq. (3.36), the v'_p can be written as

$$v'_p = \sqrt{\frac{2GMr_a}{r_p(r'_a + r_p)}} \tag{3.44}$$

$$v'_p = 10.327 \text{ km/s} \tag{3.45}$$

Now the boost required Δv at this position would be

$$\Delta v = v'_p - v_p \tag{3.46}$$

$$\Delta v = 0.076 \text{ km/s} \tag{3.47}$$

(f) **[1.5 marks]** After a series of manoeuvres, *Chandrayaan-3* was placed in an elliptical orbit of (100 × 1437) km around the Moon. Here, the distances are calculated from the surface of the Moon. Calculate the change in velocity $\Delta v'$, applied at the perigee, that is required to bring *Chandrayaan-3* from this elliptical orbit to a circular orbit at a distance of 100 km from the surface of the Moon. For this part, assume that *Chandrayaan-3* is only under the influence of the Moon's gravitational field.

Solution: Let the apogee distance be $r_{ma} = R_m + 1437$ km and perigee distance be $r_{mp} = R_m + 100$ km The velocity of the satellite at perigee, when it is in the elliptic orbit, is v_{mp} and the velocity when it is in the circular orbit is v_m we know that

$$v_{mp} = \sqrt{\frac{2GM_M r_{ma}}{r_{mp}(r_{ma}+r_{mp})}} \tag{3.48}$$

$$v_{mp} = 1.838 \text{ km/s} \tag{3.49}$$

Similarly here apogee distance be $r_{ma} = R_m + 100$ km and perigee distance be $r_{mp} = R_m + 100$ km. Let $r_m = r_{ma} = r_{mp}$

$$v_m = \sqrt{\frac{2GM_m r_m}{r_m(r_m+r_m)}} \tag{3.50}$$

$$v_m = 1.633 \text{ km/s} \tag{3.51}$$

$$\tag{3.52}$$

The boost required $\Delta v'$

$$\Delta v' = v_m - v_{mp} \tag{3.53}$$

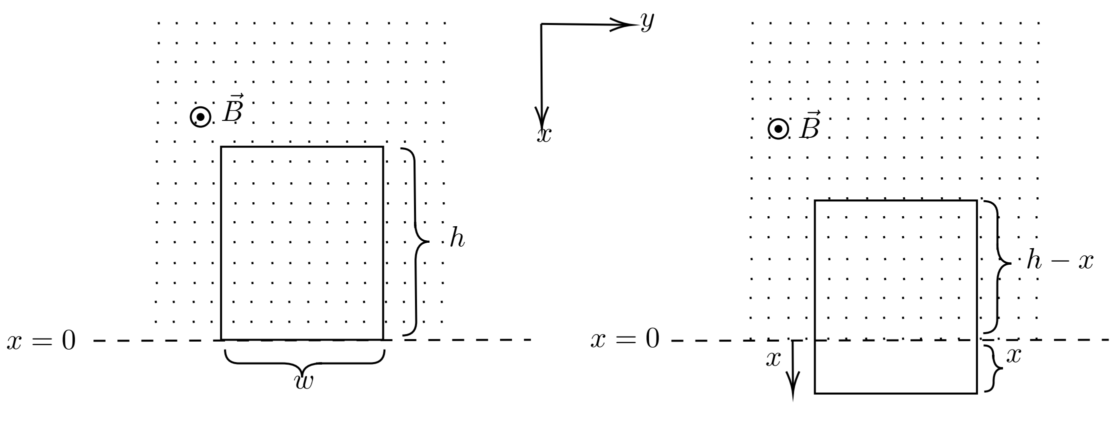
$$\Delta v' = -0.205 \text{ km/s} \tag{3.54}$$

4. **Mag-Grav Tussle**

A rectangular conducting loop of mass m, width w, length h, and self inductance L is held in the vertical x-y plane with its bottom edge along the y-axis (see figure on the left below). In this problem take the resistance of the loop to be zero. A uniform magnetic field $\vec{B}$ is applied horizontally as shown in the figure such that

$$\begin{aligned} \vec{B} &= B\hat{k} \quad \text{for } x \le 0 \\ &= 0 \quad \text{for } x > 0 \end{aligned}$$

The loop is released from rest at time $t = 0$ and descends under gravity (see the figure to the right below). The acceleration due to gravity g is in $+x$ direction.



(a) (b)

(a) **[5 marks]** Obtain $x(t)$, the position of the bottom edge of the loop at time t, in terms of relevant variables.

Solution:

$$m\ddot{x} = mg - BIw \tag{4.1}$$
$$\phi = Bw(h-x) + LI \tag{4.2}$$
$$-IR = \dot{\phi} = -Bw\dot{x} + L\dot{I} \tag{4.3}$$

Since $R = 0$ Implies

$$\dot{I} = \frac{Bw\dot{x}}{L} \tag{4.4}$$

Differentiating equation of motion with respect to t

$$m\ddot{v} = -Bw\dot{I} \tag{4.5}$$
$$\ddot{v} = \frac{-B^2w^2\dot{x}}{mL} \tag{4.6}$$
$$= -\omega_0^2 v \tag{4.7}$$

where

$$\omega_0^2 = \frac{B^2w^2}{mL} \tag{4.8}$$

The solution to v is

$$v = A\cos\omega_0 t + D\sin\omega_0 t \tag{4.9}$$
$$\dot{v} = -A\omega_0\sin\omega_0 t + D\omega_0\cos\omega_0 t \tag{4.10}$$
$$\ddot{v} = -A\omega_0^2\cos\omega_0 t + D\omega_0^2\sin\omega_0 t \tag{4.11}$$

Applying boundary conditions at $t = 0$, $\dot{v} = g$, $v = 0$ which implies that

$$D = \frac{g}{\omega_0} \tag{4.12}$$
$$A = 0 \tag{4.13}$$

Hence,

$$v = \frac{g}{\omega_0}\sin\omega_0 t \tag{4.14}$$
$$x = -\frac{g}{\omega_0^2}\cos\omega_0 t + C \tag{4.15}$$

at $t = 0$, $x = 0$, which implies that $C = \frac{g}{\omega_0^2}$

Hence,

$$x = \frac{g}{\omega_0^2}(1 - \cos\omega_0 t) \tag{4.16}$$

(b) **[6 marks]** Imagine different possible scenarios for the nature of motion of the loop and plot $x(t)$ for each.

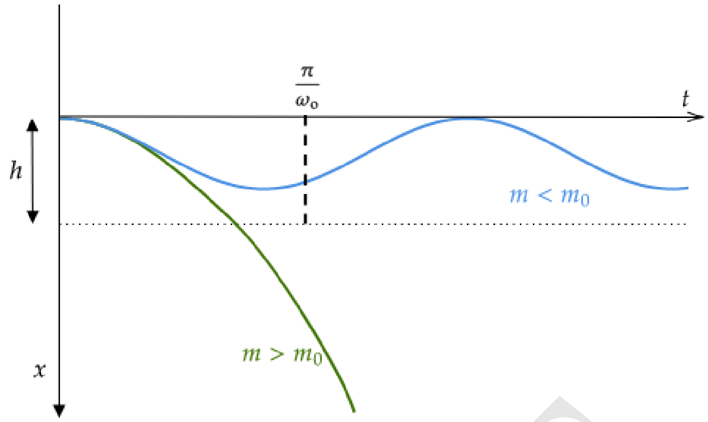
Solution: We found that

$$x = \frac{g}{\omega_0^2}(1 - \cos\omega_0 t) \tag{4.17}$$

The frequency of the oscillation is inversely proportional to m, and the amplitude increases with m. When the loop oscillates, the amplitude is

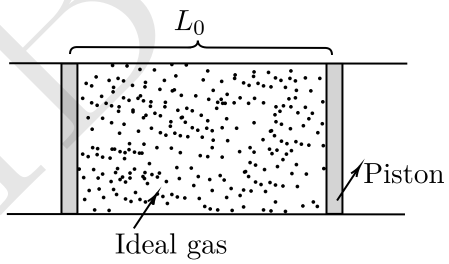
$$x_m = 2\frac{g}{\omega_0^2} = 2\frac{gmL}{B^2w^2} \tag{4.18}$$

We can take three limiting cases, $m > m_0$, and $m < m_0$, where $m_0 = hB^2w^2/2gL$.



For $m < m_0$, the loop oscillates. For $m > m_0$, the loop will come out of the magnetic field quicker it falls under gravity. Similarly, any suitable inequality involving g, L, h, B, m, and w, which distinguishes the above two cases correctly, will be considered.

5. **Thermal Tussle**
Consider a horizontal insulated cylindrical tube of very large length. Two identical insulated pistons, each of mass $M = 0.2$ kg are fitted within the tube separated by a length $L_0 = 1$ m. The space between the two pistons is filled with one mole of (ideal) helium gas, initially at temperature $T_0 = 300$K. The external pressure, everywhere outside the pistons and tube, is zero.



Initially, the pistons are held in place by an external mechanism. At time $t = 0$, the mechanism is released and the pistons move without friction and the process is quasistatic initially. Assume that the gas behaves ideally throughout. Let C_p and C_v be the specific heats of the gas at constant pressure and volume respectively. Also, $\gamma = C_p/C_v = 5/3$.

(a) **[6 marks]** Determine the velocity (v_p) of each piston in terms of the gas temperature T and other relevant variables. At what temperature (T_c), is the process no longer quasistatic? Calculate T_c.

Solution: Given the initial temperature of the system to be T_0, the initial energy of the system is C_vT_0. When the piston starts moving, from the work-energy theorem, the energy of the system is

$$\frac{1}{2}Mv_1^2 + \frac{1}{2}Mv_2^2 + C_vT = C_vT_0 \tag{5.1}$$

Also from conservation momentum, we have $v_2 = -v_1 = v$.

From above equation, we get

$$Mv^2 + C_v T = C_v T_0 \tag{5.2}$$

$$v^2 = \frac{C_v}{M}(T_0 - T) \tag{5.3}$$

$$v = \sqrt{\frac{C_v}{M}(T_0 - T)} \tag{5.4}$$

For the process to be quasistatic and adiabatic the piston's velocity cannot be greater than rms velocity of the gas.

$$v < v_{\text{rms}} \tag{5.5}$$

$$\sqrt{\frac{C_v}{M}(T_0 - T)} < \sqrt{\frac{3RT}{m}} \tag{5.6}$$

where m is molar mass of the gas. Solving the above equation, we get

$$T > \frac{C_v T_0 m}{3RM + mC_v} \approx 3\text{K} \tag{5.7}$$

Below this temperature, the piston's velocity exceeds rms velocity, which indicates that the piston moves very rapidly. This is where the quasi-static limit will breaks down. For the estimation purpose, we can also take the average velocity or the most probable velocity and the corresponding limit would be 3.5K and 4.4K respectively.

(b) **[4 marks]** From here, we restrict our analysis only to the quasistatic regime of the process. We define $u = T/T_0$. Obtain the relation between u and t in the following form

$$t = f(u)$$

You may leave the answer in terms of a suitable integral involving L_0, M and other variables.

Solution: Since the process adiabatic.

$$T_1 V_1^{\gamma-1} = T_2 V_2^{\gamma-1} \tag{5.8}$$

Which implies

$$TL^{\gamma-1} = T_0 L_0^{\gamma-1} \tag{5.9}$$

To express the temperature as a function of time, Differentiating Eq. (5.8) w.r.t t, we get

$$L^{\gamma-1}\frac{dT}{dt} + T(\gamma-1)L^{\gamma-2}\frac{dL}{dt} = 0 \tag{5.10}$$

From Eq. (5.8), we get

$$L = \left(\frac{T_0 L_0^{(\gamma-1)}}{T}\right)^{\frac{1}{\gamma-1}} \tag{5.11}$$

Also

$$\frac{dL}{dt} = 2v = 2\sqrt{\frac{C_v}{M}(T_0 - T)} \tag{5.12}$$

Substituting Eq. (5.11) and Eq. (5.12) into Eq. (5.10), we get

$$\frac{T_0 L_0^{(\gamma-1)}}{T}\frac{dT}{dt} + T(\gamma-1)\left(\frac{T_0 L_0^{(\gamma-1)}}{T}\right)^{\left(\frac{\gamma-2}{\gamma-1}\right)} 2\sqrt{\frac{C_v}{M}(T_0 - T)} = 0 \tag{5.13}$$

For Mono atomic gas $\gamma = 5/3$, hence above equation becomes

$$\frac{T_0 L_0^{2/3}}{T}\frac{dT}{dt} + T\frac{2}{3}\left(\frac{T_0 L_0^{2/3}}{T}\right)^{-1/2} 2\sqrt{\frac{C_v}{M}(T_0 - T)} = 0 \tag{5.14}$$

Rearranging above equation, we get,

$$dt = -\frac{1}{B_1}\frac{dT}{T^{5/2}(T_0 - T)^{1/2}} \tag{5.15}$$

Where $B_1 = \dfrac{2\sqrt{2/3}\sqrt{N_A k/M}}{L_0 T_0^{3/2}}$ Integrating above equation, we get

$$\int_0^t dt = -\int \frac{1}{B_1}\frac{\frac{dT}{T_0}}{(\frac{T}{T_0})^{5/2}T_0^2(1 - \frac{T}{T_0})^{1/2}} \tag{5.16}$$

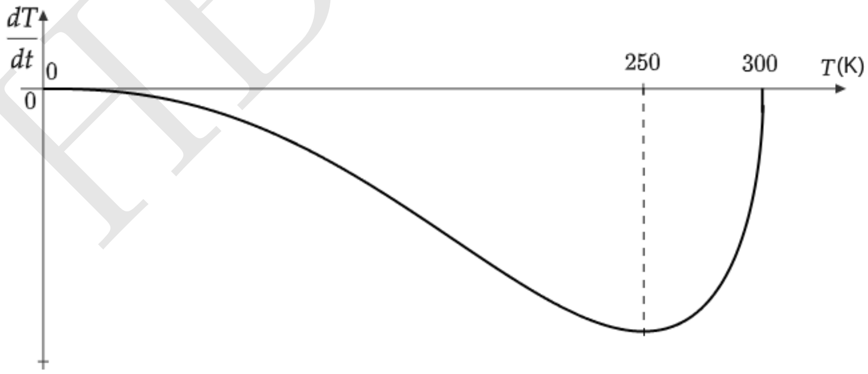
Let $u = T/T_0$, then above integral becomes

$$\int_0^t dt = -\int_{u_0}^{u} \frac{1}{B_1}\frac{du}{u^{5/2}T_0^2(1-u)^{1/2}} \tag{5.17}$$

$$t = -\frac{1}{B_1 T_0^2}\int_{u_0}^{u}\frac{du}{u^{5/2}(1-u)^{1/2}} \tag{5.18}$$

(c) [**4 marks**] Qualitatively plot the rate of change of temperature (dT/dt) vs T. Mark any significant point(s) on the temperature axis in the plot.

Solution: Temperature decreases over time. From Eq. (5.15), it is evident that the derivative of the temperature function is always negative. Additionally, at $t = 0$ and $t = 300$ K, $dT/dt = 0$. The function exhibits an extremum at $T = 250$ K. These details are illustrated in the figure below.



(d) [**4 marks**] At what time t does the temperature T of the gas reach 20K? What is the piston velocity (v_p) at this point?

Solution: The integration from Eq. (5.18) can be solved by substituting $u = \cos^2\theta$ and using boundary conditions as $u = 1$ for $T = T_0$, we get

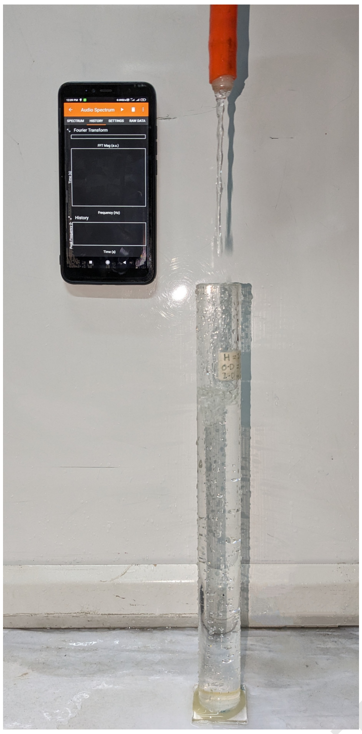
$$\left(\frac{1-u}{u}\right)^{3/2} + 3\left(\frac{1-u}{u}\right)^{1/2} - \frac{3B_1 T_0^2 t}{2} = 0 \tag{5.19}$$

Using above equation, for $n = 1$ moles and $L_0 = 1$ m, the temperature reaches 20K after 0.232s.
From Eq. (5.4) The piston's velocity at this point is 132.1 m/s.

6. **Sonic Sleuth**

During her summer vacation, Dheera decides to carry out a smartphone based experiment. She utilizes a smartphone's frequency sensor that can measure the frequency of the audio signal it receives. She takes a long cylindrical tube closed at one end. This tube has a length of $L = 30.0\,\text{cm}$ and an inner diameter of $d = 2.45\,\text{cm}$. Dheera starts filling the tube with water, which is dripping from a tap at a constant rate Q (measured in milliliters per second (mL/s)).

Dheera positions her smartphone near the open end of the tube to measure the frequency of the sound emitted as water fills the tube. An app on the phone captures a range of frequencies in the recorded audio at any given time. At randomly chosen values of time t, one of the frequencies at that time is shown in the following table.



Time t (s)	Frequency f (Hz)	Time t (s)	Frequency f (Hz)
5.0	915	36.0	434
7.6	320	39.6	481
16.2	345	41.9	500
16.7	1008	42.5	1454
20.9	360	51.1	1618
25.7	1148	51.6	574
28.9	1196	56.1	1782
31.5	410	60.2	680
33.3	1290	66.3	820

Help her to analyse the experiment.

(a) **[3 marks]** Derive the expression for the velocity of sound c_s in terms of f, t, and constants.

Solution: The frequency of the sound in the tube is determined by the formula:

$$f = \frac{nc_s}{4(h + 0.3d)} \tag{6.1}$$

Here, h and d represent the length of the air column and the diameter of the tube, respectively. The variable n is an odd integer representing the fundamental, third, fifth harmonics, and so on. The speed of sound is denoted by c_s, and the term $0.3d$ in the denominator accounts for the end correction in the tube.

When water falls at a constant rate (Q), it creates the disturbances in the air column of the tube. These disturbances travel as sound waves through the air column, which we detect and analyze. If the length of the tube is L, the equation (6.1) can be modified as:

$$c_s = \frac{f4(L - (\frac{Q}{A})t) + 0.3d)}{n} \tag{6.2}$$

Here, $A = \pi d^2/4$ represents the area of the base, and n is an odd integer representing the fundamental, third, fifth harmonics, and so on.

(b) **[8 marks]** Choose a pair of suitable variables and plot a linear graph. Specify the axis labels. Obtain the speed of sound c_s and the rate Q from this plot.

Solution: Since the height of the water level varies linearly with time, we expect the frequency of the sound to increase with time. By linearizing the equation (6.1), we get:

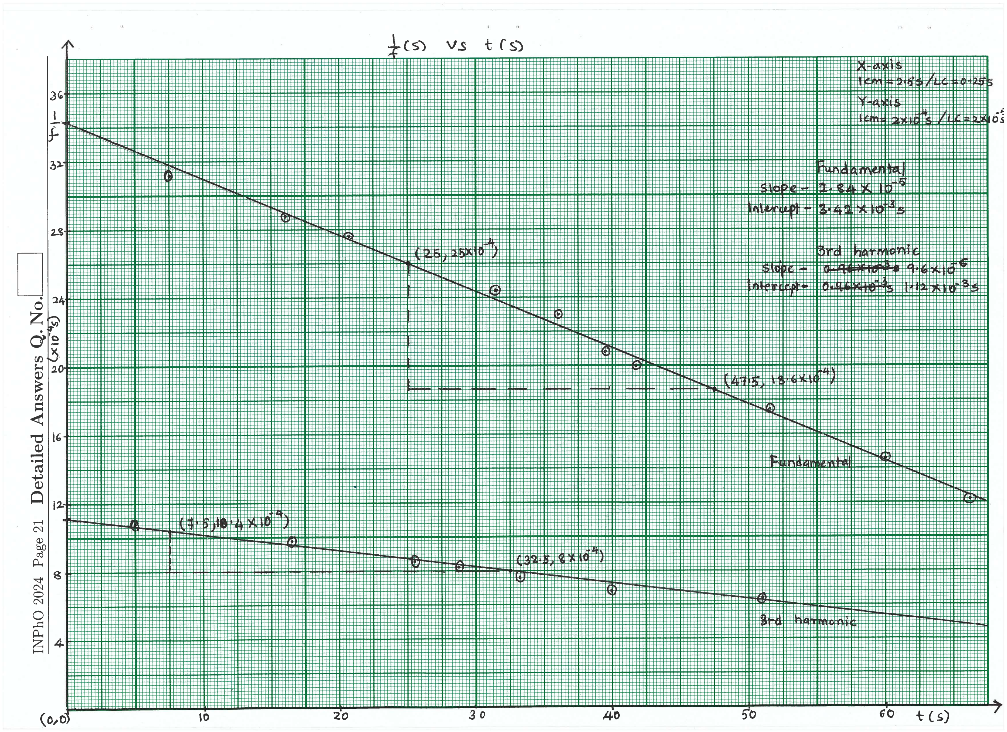
$$f = \frac{nc_s}{4(L - (\frac{Q}{A})t) + 0.3d) } \tag{6.3}$$

$$\frac{1}{f} = -\frac{4(\frac{Q}{A})t}{nc_s} + \frac{4(L + 0.3d)}{nc_s} \tag{6.4}$$

Plotting the relationship between $1/f$ and t will yield a linear graph. The values for the plot are as follows.

Fundamental			3rd Harmonic		
Time(s)	Frequency(Hz)	1/f (s)	Time(s)	Frequency(Hz)	1/f (s)
7.6	320	0.00313	5	915	0.00109
16.2	345	0.00290	16.7	1008	0.00099
20.9	360	0.00278	25.7	1148	0.00087
31.5	410	0.00244	28.9	1196	0.00084
36	434	0.00230	33.3	1290	0.00078
39.6	481	0.00208	42.5	1454	0.00069
41.9	500	0.00200	51.1	1618	0.00062
51.6	574	0.00174	56.1	1782	0.00056
60.2	680	0.00147			
66.3	820	0.00122			

The presence of two distinct straight lines in the graph indicates the existence of two harmonic frequencies in the dataset. Upon examining the ratio of these frequencies, it becomes evident that these two lines correspond to $n = 1$ and $n = 3$. The intercept and slope of the graph can be used to determine the speed of sound and the rate of water filling, respectively.



	Fundamental	3rd Harmonic
Slope	-2.84×10^{-5}	-1.12×10^{-5}
Intercept	3.42×10^{-3}	1.12×10^{-3}
C_s	359 m/s	366 m/s
Q	1.20 mL/s	1.24 mL/s

****** END OF THE QUESTION PAPER ******

Space for rough work — will NOT be submitted for evaluation