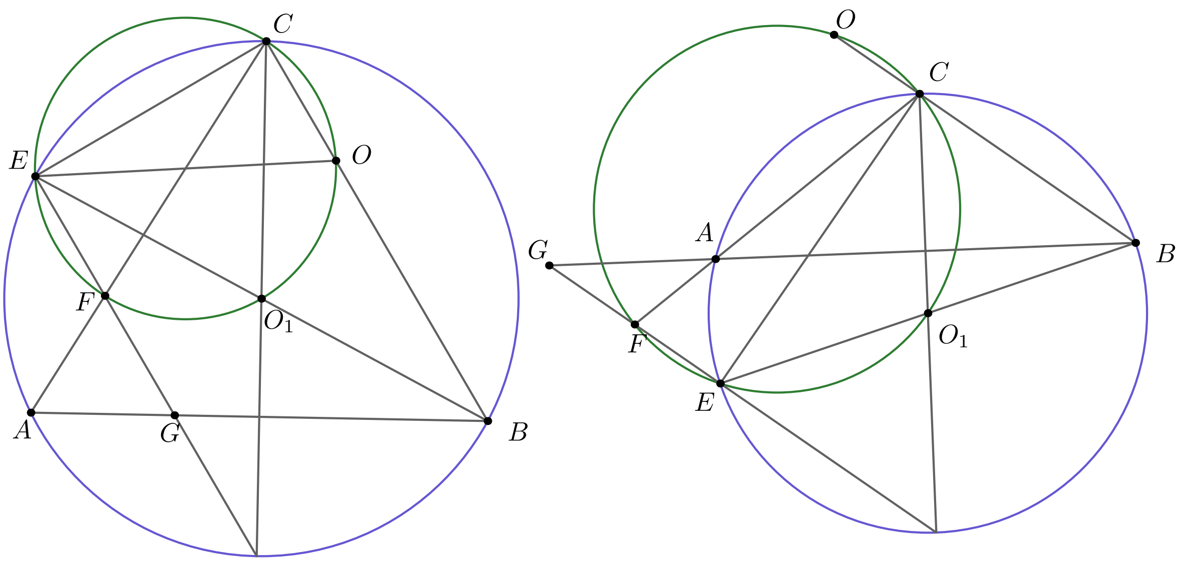
INMO 2024

Official Solutions

Problem 1. In triangle ABC with $CA = CB$, point E lies on the circumcircle of ABC such that $\angle ECB = 90^\circ$. The line through E parallel to CB intersects CA in F and AB in G. Prove that the centre of the circumcircle of triangle EGB lies on the circumcircle of triangle ECF.

Solution 1.



We have $FG = FA$ since FG is parallel to BC. But also $\triangle GAE$ is a right angle triangle. Thus, if F' is the midpoint of GE, then $\angle GAF = \angle FGA = \angle F'GA = \angle GAF'$ which implies $F \equiv F'$. Thus, F is the midpoint of GE.

If O is the circumcenter of $\triangle EBG$, then

$$\angle FOE = \angle GBE = \angle ABE = \angle ACE = \angle FCE.$$

Thus, we get $\angle FOE = \angle FCE$ as desired. □

Solution 2. ($\angle BCA$ acute case) Let O_1 be the circumcenter of $\triangle ABC$, O be the circumcenter of $\triangle EBG$ and ω be the circumcircle of $\triangle ECF$.

First, show that F is the midpoint of EG as in Solution 1. Next, we show that O_1 lies on ω. This follows from

$$\angle EO_1C = 2\angle EBC = 2\angle O_1BC = 2\angle BCO_1 = \angle BCA = \angle EFC.$$

Now, O_1 is the midpoint of EB and F is the midpoint of EG, therefore the homothety at E with ratio $1/2$ takes $\triangle EGB$ to $\triangle EFO_1$. Thus, it takes O, the circumcenter of $\triangle EGB$, to the circumcenter of $\triangle EFO_1$, thus proving that the midpoint of EO is the center of ω. This immediately implies that O lies on ω. □

Remark.

- There are two configurations possible in the above problem, one for C acute and one for obtuse. One may replace all the angles in the above solutions by *directed angles* to obtain a solution which remains invariant in all configurations.
- There is a direct solution by first proving that O is on BC by calculating $\angle EBO$ and then proving that $\angle OCF = \angle OEF$. However, this needs more care for handling both configurations.

Problem 2. All the squares of a 2024×2024 board are coloured white. In one move, Mohit can select one row or column whose every square is white, choose exactly 1000 squares in this row or column, and colour all of them red. Find the maximum number of squares that Mohit can colour red in a finite number of moves.

Solution. Let $n = 2024$ and $k = 1000$. We claim that the maximum number of squares that can be coloured in this way is $k(2n - k)$, which evaluates to 3048000.

Indeed, call a row/column *bad* if it has at least one red square. After the first move, there are exactly $k + 1$ bad rows and columns: if a row was picked, then that row and the k columns corresponding to the chosen squares are all bad. Any subsequent move increases the number of bad rows/columns by at least 1. Since there are only $2n$ rows and columns, we can make at most $2n - (k + 1)$ moves after the first one, and so at most $2n - k$ moves can be made in total. Thus we can have at most $k(2n - k)$ red squares.

To prove this is achievable, let's choose each of the n columns in the first n moves, and colour the top k cells in these columns. Then, the bottom $n - k$ rows are still uncoloured, so we can make $n - k$ more moves, colouring $k(n + n - k)$ cells in total. □

Problem 3. Let p be an odd prime number and a, b, c be integers so that the integers

$$a^{2023} + b^{2023}, \quad b^{2024} + c^{2024}, \quad c^{2025} + a^{2025}$$

are all divisible by p. Prove that p divides each of a, b, and c.

Solution 1. Set $k = 2023$. If one of a, b, c is divisible by p, then all of them are. Indeed, for example, if $p \mid a$, then $p \mid a^k + b^k$ implies $p \mid b$, and then $p \mid b^{k+1} + c^{k+1}$ implies $p \mid c$. The other cases follow similarly.

So for the sake of contradiction assume none of a, b, c is divisible by p. Then

$$a^{k(k+2)} \equiv (a^k)^{k+2} \equiv (-b^k)^{k+2} \equiv -b^{k(k+2)} \pmod p$$

and

$$a^{k(k+2)} \equiv (a^{k+2})^k = (-c^{k+2})^k \equiv -c^{k(k+2)} \pmod p.$$

So $b^{k(k+2)} \equiv c^{k(k+2)} \pmod p$. But then

$$c^{k(k+2)} \cdot c \equiv c^{(k+1)^2} \equiv (-b^{k+1})^{k+1} \equiv b^{(k+1)^2} \equiv b^{k(k+2)} \cdot b \equiv c^{k(k+2)} \cdot b \pmod p$$

which forces $b \equiv c \pmod p$. Thus

$$0 \equiv b^{k+1} + c^{k+1} = 2b^{k+1} \pmod p$$

implying $p \mid b$, a contradiction. Thus the proof is complete. □

Solution 2. As before, we may assume p divides none of a, b, and c and set $k = 2023$. Then

$$a^k \equiv -b^k \pmod p$$
$$b^{k+1} \equiv -c^{k+1} \pmod p$$
$$c^{k+2} \equiv -a^{k+2} \pmod p$$

and multiplying these three equations yields $a^k b^{k+1} c^{k+2} \equiv -b^k c^{k+1} a^{k+2} \pmod p$. By cancelling the factor $a^k b^k c^{k+1}$, we get $a^2 \equiv -bc \pmod p$. Now

$$p \mid a^k + b^k \implies a^{4k} \equiv b^{4k} \pmod p \implies c^{2k} \equiv b^{2k} \pmod p$$

so

$$p \mid b^{k+1} + c^{k+1} \implies b^{2(k+1)} \equiv c^{2(k+1)} \pmod p \implies b^2 \equiv c^2 \pmod p$$

so either $b \equiv c \pmod p$ or $b \equiv -c \pmod p$. In the latter case, $a^2 \equiv c^2 \pmod p$ so $a \equiv c \pmod p$ or $a \equiv -c \pmod p$. In any case, two out of $\{a, b, c\}$ are the same mod p, so one of the equations gives $p \mid 2x^y$ where $x \in \{a, b, c\}$ and $y \in \{k, k+1, k+2\}$, hence p odd implies $p \mid x$ so $p \mid abc$, the desired contradiction. □

Solution 3. We have

$$a^{2023} \equiv -b^{2023} \pmod p \quad (1)$$
$$b^{2024} \equiv -c^{2024} \pmod p \quad (2)$$
$$c^{2025} \equiv -a^{2025} \pmod p \quad (3)$$

Thus,

$$\begin{aligned} a^{2023\cdot 2024\cdot 2025} &\equiv b^{2023\cdot 2024\cdot 2025} \pmod p \quad \text{by } (1) \\ &\equiv -c^{2023\cdot 2024\cdot 2025} \pmod p \quad \text{by } (2) \\ &\equiv -a^{2023\cdot 2024\cdot 2025} \pmod p \quad \text{by } (3) \end{aligned}$$

Thus, $p \mid 2 \cdot a^{2023\cdot 2024\cdot 2025}$ and hence $p \mid a$ since p is odd. Now, finish as before. □

Remark.

- Solution 3 is the shortest, and seems to crucially relies on 2024 being even but it can be modified to always work. In particular, instead of raising to power $k(k+1)(k+2)$, we raise it to $\operatorname{lcm}(k, k+1, k+2)$. This method even works if we have a longer chain of equations and more variables i.e. $p | a_i^{k+i} + a_{i+1}^{k+i}$ for all i in $0, \cdots, n-1$ and $a_n = a_0$.
- There are other possible approaches: one using primitive roots and another using orders. These proceed by considering the highest power of 2 dividing the exponents of the primitive root or order of $ab^{-1} \bmod p$ respectively.

Problem 4. A finite set S of positive integers is called *cardinal* if S contains the integer $|S|$, where $|S|$ denotes the number of distinct elements in S. Let f be a function from the set of positive integers to itself, such that for any cardinal set S, the set $f(S)$ is also cardinal. Here $f(S)$ denotes the set of all integers that can be expressed as $f(a)$ for some a in S. Find all possible values of $f(2024)$.
Note: As an example, $\{1, 3, 5\}$ is a cardinal set because it has exactly 3 distinct elements, and the set contains 3.

Solution 1. The possible values are $1, 2$, and 2024.

Construction. The function $f(x) = 1$ for all $x \in \mathbb{N}$ works. Also, $f(x) = 1$ for all $x \neq 2024$ and $f(2024) = 2$, works. Finally, $f(x) = x$ for all $x \in \mathbb{N}$ works as well.

It remains to show these are the only possible values for $f(2024)$.

Proof. Denote $\text{Im}(f) = \{f(x) \mid x \in \mathbb{N}\}$. The cardinal set $\{1\}$ gives $f(1) = 1$. Consider the following two cases:

- $\text{Im}(f)$ is unbounded. Fix any $n \in \mathbb{N}$, with $n > 1$. Pick $n-1$ distinct integers $k_1, \ldots, k_{n-1}$ such that $f(k_i) \notin \{n, f(n)\}$ and $f(k_i)$ are all pairwise distinct, for $1 \leq i < n$. Then $\{n, k_1, \ldots, k_{n-1}\}$ is a cardinal set. Then $\{f(n), f(k_1), \ldots, f(k_{n-1})\}$ is a cardinal set with n distinct elements, so n lies in this set, hence $f(n) = n$. This gives the identity function.
- $\text{Im}(f)$ is bounded. Suppose $f(x) \leq M$ for all $x \in \mathbb{N}$ and some integer $M > 0$.

  Claim. For any integer a satisfying $1 \leq a \leq M$, if there are infinitely many integers $n \in \mathbb{N}$ such that $f(n) = a$, then $a = 1$.

  Proof. Let $b > 1$ be one of the integers with $f(b) = a$. Consider $b - 1$ other integers $c_1, \ldots, c_{b-1}$, such that $f(c_i) = a$ for $1 \leq i < b$, and c_i are all pairwise distinct. Then $\{b, c_1, \ldots, c_{b-1}\}$ is a cardinal set, so the image set, which consists of the singleton $\{a\}$ is cardinal, hence $a = 1$. □

  So for every $2 \leq m \leq M$, there are only finitely many integers x such that $f(x) = m$. Thus, there exists an integer $N > 1$ such that for all $n \geq N$, $f(n) = 1$. Now for every $1 < l < N$, consider the cardinal set $\{l, N+1, N+2, \ldots, N+l-1\}$. Then the image set consists of $\{1, f(l)\}$, which can be cardinal only when $f(l) = 1$ or $f(l) = 2$.

By the above reasoning, $f(2024)$ can only be $1, 2$, or 2024, each of which occurs as an example. □

Solution 2. We present a second proof of the fact that the proposed values are the only possibilities. Considering the singleton cardinal set $\{1\}$, we see that $f(1) = 1$. The cardinal set $\{1, 2\}$ gets mapped to $\{1, f(2)\}$, so $f(2)$ must be 2 or 1.

Case 1. Suppose $f(2) = 1$. Now $\{2, 2024\}$ is a cardinal set, and therefore so is $\{1, f(2024)\}$. This means $f(2024)$ is 1 or 2.

Case 2. Suppose $f(2) = 2$. The cardinal set $f(\{1, 2, 3\}) = \{1, 2, f(3)\}$ shows that $f(3) \in \{1, 2, 3\}$, but the cardinal set $f(\{2, 3\}) = \{2, f(3)\}$ proves $f(3)$ cannot be 2. Thus there are two sub-cases:

2.1. $f(3) = 1$. Then the set $\{1, 3, 2024\}$ is cardinal, hence so is $\{1, f(2024)\}$, implying, as before, $f(2024) \in \{1, 2\}$.

2.2. $f(3) = 3$. In this case, we show via induction that $f(n) = n$ for all $n \in \mathbb{N}$.

The base cases $n = 1, 2, 3$ are already known. Now consider $n \geq 4$, and assume $f(k) = k$ for all $k < n$. Consider the cardinal $f(\{1, 2, \ldots, n\}) = \{1, 2, \ldots, n-1, f(n)\}$ which implies $f(n) \in \{1, 2, \ldots, n\}$.

However, consider the $n - 1$-element cardinal set $\{1, 2, \ldots, n\} \setminus \{n - 2\}$. For its image to be cardinal $f(n)$ cannot equal any number in $\{1, 2, \ldots, n - 1\} \setminus \{n - 2\}$; else its cardinality would be $n - 2$, which isn't in the set. So $f(n) \in \{n - 2, n\}$.

Finally, consider the $n - 2$-element set $\{1, 2, \ldots, n\} \setminus \{n - 1, n - 3\}$. If $f(n) = n - 2$, its image would only have $n - 3$ elements, and thus would not be cardinal. So we conclude that $f(n) = n$ and the induction is complete. In particular, $f(2024) = 2024$.

Thus the only possible values of $f(2024)$ are $1, 2$, and 2024. □

Remark.

- There are many ways to finish Solution 2 after reaching (2.2):
  1. Induct on n and get $1 \leq f(n) \leq n$ as before. Now, note that $f(n) \neq 1$ for $n > 3$ by considering $\{1, 3, n\}$, and then if $f(n) < n$ we get a contradiction on considering $\{1, 2, \ldots, f(n) - 2, f(n), n\}$.
  2. Show that $f(n) \neq 1, 2, 3$ for $n > 3$ by considering $\{2, n\}$ and $\{1, 3, n\}$. Then prove that f is *injective* by considering $\{3, n, m\}$ if $n > m > 3$. Now, finish by induction and considering $\{1, 2, \ldots, n\}$.
  3. Suppose n_0 is the smallest integer so that $f(n_0) \neq n_0$ for some $n_0 > 3$, and let t be the smallest value achieved by $f(n)$ for $n \geq n_0$. Let $f(m) = t$. Then $t \neq 1, 2, 3$ and $t < n_0 \leq m$ as before. And now consider $\{t, m, m + 1, \ldots, m + t - 2\}$ to get that f must take on a value smaller than t.

Problem 5. Let points A_1, A_2, and A_3 lie on the circle Γ in counter-clockwise order, and let P be a point in the same plane. For $i \in \{1, 2, 3\}$, let τ_i denote the counter-clockwise rotation of the plane centred at A_i, where the angle of the rotation is equal to the angle at vertex A_i in $\triangle A_1A_2A_3$. Further, define P_i to be the point $\tau_{i+2}(\tau_i(\tau_{i+1}(P)))$, where indices are taken modulo 3 (*i.e.*, $\tau_4 = \tau_1$ and $\tau_5 = \tau_2$).

Prove that the radius of the circumcircle of $\triangle P_1P_2P_3$ is at most the radius of Γ.

Solution 1. Fix an index $i \in \{1, 2, 3\}$. Let D_1, D_2, D_3 be the points of tangency of the incircle of triangle $\triangle A_1A_2A_3$ with its sides A_2A_3, A_3A_1, A_1A_2 respectively.

The key observation is that given a line ℓ in the plane, the image of ℓ under the mapping $\tau_{i+2}(\tau_i(\tau_{i+1}(\ell)))$ is a line parallel to ℓ. Indeed, ℓ is rotated thrice by angles equal to the angles of $\triangle A_1A_2A_3$, and the composition of these rotations induces a half-turn and translation on ℓ as the angles of $\triangle A_1A_2A_3$ add to $180°$. Since D_i is a fixed point of this transformation (by the chain of maps $D_i \overset{\tau_{i+1}}{\to} D_{i+2} \overset{\tau_i}{\to} D_{i+1} \overset{\tau_{i+2}}{\to} D_i$), we conclude that the line $\overline{PD_i}$ maps to the line $\overline{P_iD_i}$. But the two lines are parallel and both of them pass through D_i hence they must coincide, so D_i lies on $\overline{PP_i}$. Further, each rotation preserves distances, hence P_i is the reflection of P in D_i.

In other words, the triangle $P_1P_2P_3$ is obtained by applying a homothety with ratio 2 and center P to the triangle $D_1D_2D_3$. Thus, the radius of the circumcircle of $\triangle P_1P_2P_3$ is twice the radius of the circumcircle of $\triangle D_1D_2D_3$, *i.e.*, twice the radius of the incircle of $\triangle A_1A_2A_3$, which is known to be at most the radius of the circumcircle Γ.

Solution 2. Toss the figure on the complex plane, and let $A_1 = a$, $A_2 = b$, $A_3 = c$ without loss of generality. Let the angles of the triangle at A_1, A_2, A_3 be denoted by A, B, C.

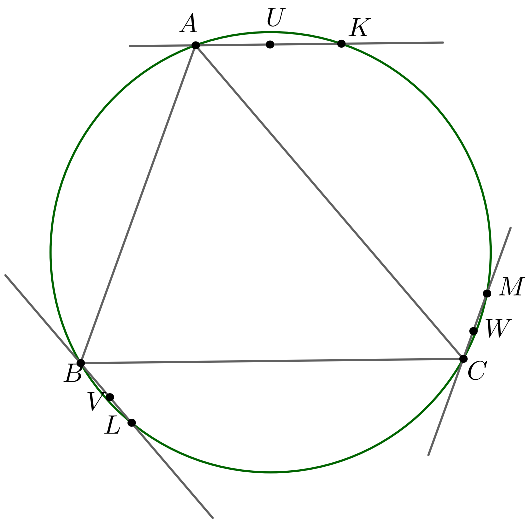
Now, for any complex number z, the rotation at z_0 with angle θ counterclockwise sends z to $(z - z_0)e^{i\theta} + z_0$.

Therefore, one computes that

$$\begin{aligned}\tau_{321}(z) = \tau_3(\tau_1(\tau_2(z))) &= \tau_3(\tau_1((z-b)e^{iB} + b)) \\ &= \tau_3(ze^{i(A+B)} + be^{iA}(1 - e^{iB}) + a(1 - e^{iA})) \\ &= -z + b + c + be^{i(A+C)} + ae^{iC} - ae^{i(A+C)} - ce^{iC}\end{aligned}$$

Thus, $\tau_{312}(z)+z$ is independent of z. Similarly $\tau_{123}(z)+z$ and $\tau_{231}(z)+z$ are also independent of z. Note that adding z is the same as translation by z, hence we have shown that the circumradius of $\triangle P_1P_2P_3$ is independent of P.

Thus, it suffices to prove the result for $z = z_0 = a + b + c$. Let $U = -\tau_{312}(z_0), V = -\tau_{123}(z_0), W = -\tau_{231}(z_0)$. So, it is enough to prove that the circumradius of $\triangle UVW$ at most the radius of Γ.



Name the vertices A_1, A_2, A_3 as A, B, C for convenience. Let the parallel line to BC passing through A intersect Γ again at K. Similarly, define L as the second intersection of the line through B parallel to CA and finally M for C parallel to AB.

We claim that U lies on the line segment $\overline{AK}$: We have $U = a - (b-a)e^{i(A+C)} + (c-a)e^{iC}$, hence $\overrightarrow{AU}$ is parallel to $\overrightarrow{AK}$ hence U lies on the line AK. If $AB = AC$ then $U = A$, and the claim is proven. Else suppose that $AB < AC$. Then $\overrightarrow{AU}$ points towards K and $|AU| = AC - AB$, so it suffices to show that $AK > AC - AB$. But this is clear because $KCBA$ is an isosceles trapezium, so $AB = KC$, and then triangle inequality on $\triangle KAC$ to get $KA + KC > AC$.

Thus, $U \in \overline{AK}$, and similarly $V \in \overline{BL}$, $W \in \overline{CM}$. We claim that for any U, V, W on the segments AK, BL, CM respectively, the circumradius of $\triangle UVW$ is less than or equal to the radius of Γ.

Now let X, Y be two fixed points on the same side of a line ℓ. Fix a side of $\overleftrightarrow{XY}$, and let Z be a variable point on ℓ which always remains on this fixed side of $\overleftrightarrow{XY}$. Then the circumradius of $\triangle XYZ$ is minimized at the unique point Z_0 (on this fixed side of $\overleftrightarrow{XY}$) for which the circumcircle of $\triangle XYZ_0$ is tangent to ℓ and it is a increasing function as one goes further away from this unique point Z_0. Thus, the maximum circumradius of $\triangle UVW$ is achieved only if $U \in \{A, K\}$, $V \in \{B, L\}$, $W \in \{C, M\}$. For each of these, the circumradius is the radius of Γ, hence we are done. $\square$

Remark.

- The conclusion of Solution 1 used the fact that in a triangle ABC with incentre I and inradius r, and circumcentre O and circumradius R, we have the inequality $R \geq 2r$. This is called Euler's Inequality. The standard proof is that $0 \leq OI^2 = R^2 - \text{Pow}(I, (O, R)) = R^2 - 2Rr$. The last equality holds as $\text{Pow}(I, (O, R)) = IA \cdot IM$ where M is the midpoint of minor arc $\widehat{BC}$ in the circumcircle of ABC, and because $IA = \frac{r}{\sin \frac{A}{2}}$ and $IM = MB = \frac{a}{2\cos\frac{A}{2}} = \frac{2R\sin A}{2\cos\frac{A}{2}} = 2R\sin\frac{A}{2}$ by using "the trident lemma" and the double-angle sine formulas.

- After proving that the circumradius is independent of P, one can take $P = I$, for which P_1 is easily seen to be the point such that D_1 is the midpoint of IP_1. Now we again finish by Euler's Inequality.

Problem 6. For each positive integer $n \geq 3$, define A_n and B_n as

$$A_n = \sqrt{n^2+1} + \sqrt{n^2+3} + \cdots + \sqrt{n^2+2n-1},$$

$$B_n = \sqrt{n^2+2} + \sqrt{n^2+4} + \cdots + \sqrt{n^2+2n}.$$

Determine all positive integers $n \geq 3$ for which $\lfloor A_n \rfloor = \lfloor B_n \rfloor$.
Note. For any real number x, $\lfloor x \rfloor$ denotes the largest integer N such that $N \leq x$.

Solution. Let $M = n^2 + \frac{1}{2}n$.

Lemma 1. $B_n - A_n < \frac{1}{2}$.
Indeed,

$$(B_n - A_n) = \sum_{k=1}^{n} \left(\sqrt{n^2+2k} - \sqrt{n^2+2k-1}\right) = \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+2k}+\sqrt{n^2+2k-1}} < \sum_{k=1}^{n} \frac{1}{2n} = \frac{n}{2n} = \frac{1}{2}$$

proving the lemma.

Lemma 2. $A_n < M < B_n$.

Proof. Observe that

$$(A_n - n^2) = \sum_{k=1}^{n} \left(\sqrt{n^2+2k-1} - n\right) = \sum_{k=1}^{n} \frac{2k-1}{\sqrt{n^2+2k-1}+n} < \sum_{k=1}^{n} \frac{2k-1}{n+n} = \frac{n^2}{2n} = \frac{n}{2}$$

as $\sum_{k=1}^{n}(2k-1) = n^2$, proving $A_n - n^2 < \frac{n}{2}$ or $A_n < M$. Similarly,

$$(B_n - n^2) = \sum_{k=1}^{n} \left(\sqrt{n^2+2k} - n\right) = \sum_{k=1}^{n} \frac{2k}{\sqrt{n^2+2k}+n} > \sum_{k=1}^{n} \frac{2k}{(n+1)+n} = \frac{n(n+1)}{2n+1} > \frac{n}{2}$$

as $\sum_{k=1}^{n}(2k) = n(n+1)$, so $B_n - n^2 > \frac{n}{2}$ hence $B_n > M$, as desired. $\square$

By Lemma 2, we see that A_n and B_n are positive real numbers containing M between them. When n is even, M is an integer. This implies $\lfloor A_n \rfloor < M$, but $\lfloor B_n \rfloor \geq M$, which means we cannot have $\lfloor A_n \rfloor = \lfloor B_n \rfloor$.

When n is odd, M is a half-integer, and thus $M - \frac{1}{2}$ and $M + \frac{1}{2}$ are consecutive integers. So the above two lemmas imply

$$M - \frac{1}{2} < B_n - (B_n - A_n) = A_n < B_n = A_n + (B_n - A_n) < M + \frac{1}{2}.$$

This shows $\lfloor A_n \rfloor = \lfloor B_n \rfloor = M - \frac{1}{2}$.

Thus, the only integers $n \geq 3$ that satisfy the conditions are the odd numbers and all of them work. $\square$