INMO 2024

Official Solutions

Problem 1. In triangle ABC with $CA = CB$, point E lies on the circumcircle of ABC such that $\angle ECB = 90^\circ$. The line through E parallel to CB intersects CA in F and AB in G. Prove that the centre of the circumcircle of triangle EGB lies on the circumcircle of triangle ECF.

Solution. We have $FG = FA$ since FG is parallel to BC. But also $\triangle GAE$ is a right angle triangle. Thus, if F' is the midpoint of GE, then $\angle GAF = \angle FGA = \angle F'GA = \angle GAF'$ which implies $F \equiv F'$. Thus, F is the midpoint of GE.

If O is the circumcenter of $\triangle EBG$, then

$$\angle FOE = \angle GBE = \angle ABE = \angle ACE = \angle FCE.$$

Thus, we get $\angle FOE = \angle FCE$ as desired. □

Problem 2. All the squares of a 2024×2024 board are coloured white. In one move, Mohit can select one row or column whose every square is white, choose exactly 1000 squares in this row or column, and colour all of them red. Find the maximum number of squares that Mohit can colour red in a finite number of moves.

Solution. Let $n = 2024$ and $k = 1000$. We claim that the maximum number of squares that can be coloured in this way is $k(2n - k)$, which evaluates to 3048000.

Indeed, call a row/column *bad* if it has at least one red square. After the first move, there are exactly $k + 1$ bad rows and columns: if a row was picked, then that row and the k columns corresponding to the chosen squares are all bad. Any subsequent move increases the number of bad rows/columns by at least 1. Since there are only $2n$ rows and columns, we can make at most $2n - (k + 1)$ moves after the first one, and so at most $2n - k$ moves can be made in total. Thus we can have at most $k(2n - k)$ red squares.

To prove this is achievable, let's choose each of the n columns in the first n moves, and colour the top k cells in these columns. Then, the bottom $n - k$ rows are still uncoloured, so we can make $n - k$ more moves, colouring $k(n + n - k)$ cells in total. □

Problem 3. Let p be an odd prime number and a, b, c be integers so that the integers

$$a^{2023} + b^{2023}, \quad b^{2024} + c^{2024}, \quad c^{2025} + a^{2025}$$

are all divisible by p. Prove that p divides each of $a, b,$ and c.

Solution 1. Set $k = 2023$. If one of a, b, c is divisible by p, then all of them are. Indeed, for example, if $p \mid a$, then $p \mid a^k + b^k$ implies $p \mid b$, and then $p \mid b^{k+1} + c^{k+1}$ implies $p \mid c$. The other cases follow similarly.

So for the sake of contradiction assume none of a, b, c is divisible by p. Then

$$a^{k(k+2)} \equiv (a^k)^{k+2} \equiv (-b^k)^{k+2} \equiv -b^{k(k+2)} \pmod p$$

and

$$a^{k(k+2)} \equiv (a^{k+2})^k = (-c^{k+2})^k \equiv -c^{k(k+2)} \pmod p.$$

So $b^{k(k+2)} \equiv c^{k(k+2)} \pmod p$. But then

$$c^{k(k+2)} \cdot c \equiv c^{(k+1)^2} \equiv (-b^{k+1})^{k+1} \equiv b^{(k+1)^2} \equiv b^{k(k+2)} \cdot b \equiv c^{k(k+2)} \cdot b \pmod p$$

which forces $b \equiv c \pmod p$. Thus

$$0 \equiv b^{k+1} + c^{k+1} = 2b^{k+1} \pmod p$$

implying $p \mid b$, a contradiction. Thus the proof is complete. □

Solution 2. As before, we may assume p divides none of $a, b,$ and c and set $k = 2023$. Then

$$\begin{aligned} a^k &\equiv -b^k \pmod p \\ b^{k+1} &\equiv -c^{k+1} \pmod p \\ c^{k+2} &\equiv -a^{k+2} \pmod p \end{aligned}$$

and multiplying these three equations yields $a^k b^{k+1} c^{k+2} \equiv -b^k c^{k+1} a^{k+2} \pmod p$. By cancelling the factor $a^k b^k c^{k+1}$, we get $a^2 \equiv -bc \pmod p$. Now

$$p \mid a^k + b^k \implies a^{2k} \equiv b^{2k} \pmod p \implies c^k \equiv b^k \pmod p$$

so

$$p \mid b^{k+1} + c^{k+1} \implies b^{2(k+1)} \equiv c^{2(k+1)} \pmod p \implies b^2 \equiv c^2 \pmod p$$

so either $b = c \pmod p$ or $b = -c \pmod p$. In the latter case, $a^2 = c^2 \pmod p$ so $a = c \pmod p$ or $a = -c \pmod p$. In any case, two out of $\{a, b, c\}$ are the same mod p, so one of the equations gives $p \mid 2x^y$ where $x \in \{a, b, c\}$ and $y \in \{k, k+1, k+2\}$, hence p odd implies $p \mid x$ so $p \mid abc$, the desired contradiction. $\square$

Problem 4. A finite set S of positive integers is called *cardinal* if S contains the integer $|S|$, where $|S|$ denotes the number of distinct elements in S. Let f be a function from the set of positive integers to itself, such that for any cardinal set S, the set $f(S)$ is also cardinal. Here $f(S)$ denotes the set of all integers that can be expressed as $f(a)$ for some a in S. Find all possible values of $f(2024)$.
Note: As an example, $\{1, 3, 5\}$ is a cardinal set because it has exactly 3 distinct elements, and the set contains 3.

Solution 1. The possible values are $1, 2,$ and 2024.

Construction. The function $f(x) = 1$ for all $x \in \mathbb{N}$ works. Also, $f(x) = 1$ for all $x \neq 2024$ and $f(2024) = 2$, works. Finally, $f(x) = x$ for all $x \in \mathbb{N}$ works as well.

It remains to show these are the only possible values for $f(2024)$.

Proof. Denote $\text{Im}(f) = \{f(x) \mid x \in \mathbb{N}\}$. The cardinal set $\{1\}$ gives $f(1) = 1$. Consider the following two cases:

- $\text{Im}(f)$ is unbounded. Fix any $n \in \mathbb{N}$, with $n > 1$. Pick $n-1$ distinct integers $k_1, \ldots, k_{n-1}$ such that $f(k_i) \notin \{n, f(n)\}$ and $f(k_i)$ are all pairwise distinct, for $1 \leq i < n$. Then $\{n, k_1, \ldots, k_{n-1}\}$ is a cardinal set. Then $\{f(n), f(k_1), \ldots, f(k_{n-1})\}$ is a cardinal set with n distinct elements, so n lies in this set, hence $f(n) = n$. This gives the identity function.
- $\text{Im}(f)$ is bounded. Suppose $f(x) \leq M$ for all $x \in \mathbb{N}$ and some integer $M > 0$.

  Claim. For any integer a satisfying $1 \leq a \leq M$, if there are infinitely many integers $n \in \mathbb{N}$ such that $f(n) = a$, then $a = 1$.

  Proof. Let $b > 1$ be one of the integers with $f(b) = a$. Consider $b - 1$ other integers $c_1, \ldots, c_{b-1}$, such that $f(c_i) = a$ for $1 \leq i < b$, and c_i are all pairwise distinct. Then $\{b, c_1, \ldots, c_{b-1}\}$ is a cardinal set, so the image set, which consists of the singleton $\{a\}$ is cardinal, hence $a = 1$. $\square$

  So for every $2 \leq m \leq M$, there are only finitely many integers x such that $f(x) = m$. Thus, there exists an integer $N > 1$ such that for all $n \geq N$, $f(n) = 1$. Now for every $1 < l < N$, consider the cardinal set $\{l, N+1, N+2, \ldots, N+l-1\}$. Then the image set consists of $\{1, f(l)\}$, which can be cardinal only when $f(l) = 1$ or $f(l) = 2$.

By the above reasoning, $f(2024)$ can only be $1, 2,$ or 2024, each of which occurs as an example. $\square$

Solution 2. We present a second proof of the fact that the proposed values are the only possibilities. Considering the singleton cardinal set $\{1\}$, we see that $f(1) = 1$. The cardinal set $\{1, 2\}$ gets mapped to $\{1, f(2)\}$, so $f(2)$ must be 2 or 1.

Case 1. Suppose $f(2) = 1$. Now $\{2, 2024\}$ is a cardinal set, and therefore so is $\{1, f(2024)\}$. This means $f(2024)$ is 1 or 2.

Case 2. Suppose $f(2) = 2$. The cardinal set $f(\{1, 2, 3\}) = \{1, 2, f(3)\}$ shows that $f(3) \in \{1, 2, 3\}$, but the cardinal set $f(\{2, 3\}) = \{2, f(3)\}$ proves $f(3)$ cannot be 2. Thus there are two sub-cases:

2.1. $f(3) = 1$. Then the set $\{1, 3, 2024\}$ is cardinal, hence so is $\{1, f(2024)\}$, implying, as before, $f(2024) \in \{1, 2\}$.

2.2. $f(3) = 3$. In this case, we show via induction that $f(n) = n$ for all $n \in \mathbb{N}$.

The base cases $n = 1, 2, 3$ are already known. Now consider $n \geq 4$, and assume $f(k) = k$ for all $k < n$. Consider the cardinal $f(\{1, 2, \ldots, n\}) = \{1, 2, \ldots, n-1, f(n)\}$ which implies $f(n) \in \{1, 2, \ldots, n\}$.

However, consider the $n-1$-element cardinal set $\{1, 2, \ldots, n\} \setminus \{n-2\}$. For its image to be cardinal $f(n)$ cannot equal any number in $\{1, 2, \ldots, n-1\} \setminus \{n-2\}$; else its cardinality would be $n-2$, which isn't in the set. So $f(n) \in \{n-2, n\}$.

Finally, consider the $n-2$-element set $\{1, 2, \ldots, n\} \setminus \{n-1, n-3\}$. If $f(n) = n-2$, its image would only have $n-3$ elements, and thus would not be cardinal. So we conclude that $f(n) = n$ and the induction is complete. In particular, $f(2024) = 2024$.

Thus the only possible values of $f(2024)$ are $1, 2$, and 2024. $\square$

Problem 5. Let points A_1, A_2, and A_3 lie on the circle Γ in counter-clockwise order, and let P be a point in the same plane. For $i \in \{1, 2, 3\}$, let τ_i denote the counter-clockwise rotation of the plane centred at A_i, where the angle of the rotation is equal to the angle at vertex A_i in $\triangle A_1A_2A_3$. Further, define P_i to be the point $\tau_{i+2}(\tau_i(\tau_{i+1}(P)))$, where indices are taken modulo 3 (*i.e.*, $\tau_4 = \tau_1$ and $\tau_5 = \tau_2$).

Prove that the radius of the circumcircle of $\triangle P_1P_2P_3$ is at most the radius of Γ.

Solution. Fix an index $i \in \{1, 2, 3\}$. Let D_1, D_2, D_3 be the points of tangency of the incircle of triangle $\triangle A_1A_2A_3$ with its sides A_2A_3, A_3A_1, A_1A_2 respectively.

The key observation is that given a line ℓ in the plane, the image of ℓ under the mapping $\tau_{i+2}(\tau_i(\tau_{i+1}(\ell)))$ is a line parallel to ℓ. Indeed, ℓ is rotated thrice by angles equal to the angles of $\triangle A_1A_2A_3$, and the composition of these rotations induces a half-turn and translation on ℓ as the angles of $\triangle A_1A_2A_3$ add to $180°$. Since D_i is a fixed point of this transformation (by the chain of maps $D_i \overset{\tau_{i+1}}{\to} D_{i+2} \overset{\tau_i}{\to} D_{i+1} \overset{\tau_{i+2}}{\to} D_i$), we conclude that the line $\overline{PD_i}$ maps to the line $\overline{P_iD_i}$. But the two lines are parallel and both of them pass through D_i hence they must coincide, so D_i lies on $\overline{PP_i}$. Further, each rotation preserves distances, hence P_i is the reflection of P in D_i.

In other words, the triangle $P_1P_2P_3$ is obtained by applying a homothety with ratio 2 and center P to the triangle $D_1D_2D_3$. Thus, the radius of the circumcircle of $\triangle P_1P_2P_3$ is twice the radius of the circumcircle of $\triangle D_1D_2D_3$, *i.e.*, twice the radius of the incircle of $\triangle A_1A_2A_3$, which is known to be at most the radius of the circumcircle Γ.

Remark. The conclusion used the fact that in a triangle ABC with incentre I and inradius r, and circumcentre O and circumradius R, we have the inequality $R \geq 2r$. This is called Euler's Inequality. The standard proof is that $0 \leq OI^2 = R^2 - \text{Pow}(I, (O, R)) = R^2 - 2Rr$. The last equality holds as $\text{Pow}(I, (O, R)) = IA \cdot IM$ where M is the midpoint of minor arc $\widehat{BC}$ in the circumcircle of ABC, and because $IA = \frac{r}{\sin\frac{A}{2}}$ and $IM = MB = \frac{a}{2\cos\frac{A}{2}} = \frac{2R\sin A}{2\cos\frac{A}{2}} = 2R\sin\frac{A}{2}$ by using "the trident lemma" and the double-angle sine formulas.

Problem 6. For each positive integer $n \geq 3$, define A_n and B_n as

$$A_n = \sqrt{n^2+1} + \sqrt{n^2+3} + \cdots + \sqrt{n^2+2n-1},$$

$$B_n = \sqrt{n^2+2} + \sqrt{n^2+4} + \cdots + \sqrt{n^2+2n}.$$

Determine all positive integers $n \geq 3$ for which $\lfloor A_n \rfloor = \lfloor B_n \rfloor$.
Note. For any real number x, $\lfloor x \rfloor$ denotes the largest integer N such that $N \leq x$.

Solution. Let $M = n^2 + \frac{1}{2}n$.

Lemma 1. $B_n - A_n < \frac{1}{2}$.
Indeed,

$$(B_n - A_n) = \sum_{k=1}^{n} \left(\sqrt{n^2+2k} - \sqrt{n^2+2k-1}\right) = \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+2k}+\sqrt{n^2+2k-1}} < \sum_{k=1}^{n} \frac{1}{2n} = \frac{n}{2n} = \frac{1}{2}$$

proving the lemma.

Lemma 2. $A_n < M < B_n$.

Proof. Observe that

$$(A_n - n^2) = \sum_{k=1}^{n} \left(\sqrt{n^2+2k-1} - n\right) = \sum_{k=1}^{n} \frac{2k-1}{\sqrt{n^2+2k-1}+n} < \sum_{k=1}^{n} \frac{2k-1}{n+n} = \frac{n^2}{2n} = \frac{n}{2}$$

as $\sum_{k=1}^{n}(2k-1) = n^2$, proving $A_n - n^2 < \frac{n}{2}$ or $A_n < M$. Similarly,

$$(B_n - n^2) = \sum_{k=1}^{n} \left(\sqrt{n^2+2k} - n\right) = \sum_{k=1}^{n} \frac{2k}{\sqrt{n^2+2k}+n} > \sum_{k=1}^{n} \frac{2k}{(n+1)+n} = \frac{n(n+1)}{2n+1} > \frac{n}{2}$$

as $\sum_{k=1}^{n}(2k) = n(n+1)$, so $B_n - n^2 > \frac{n}{2}$ hence $B_n > M$, as desired. $\square$

By Lemma 2, we see that A_n and B_n are positive real numbers containing M between them. When n is even, M is an integer. This implies $\lfloor A_n \rfloor < M$, but $\lfloor B_n \rfloor \geq M$, which means we cannot have $\lfloor A_n \rfloor = \lfloor B_n \rfloor$.

When n is odd, M is a half-integer, and thus $M - \frac{1}{2}$ and $M + \frac{1}{2}$ are consecutive integers. So the above two lemmas imply

$$M - \frac{1}{2} < B_n - (B_n - A_n) = A_n < B_n = A_n + (B_n - A_n) < M + \frac{1}{2}.$$

This shows $\lfloor A_n \rfloor = \lfloor B_n \rfloor = M - \frac{1}{2}$.

Thus, the only integers $n \geq 3$ that satisfy the conditions are the odd numbers and all of them work. $\square$