Regional Mathematical Olympiad-2023

1. Let $\mathbb{N}$ be the set of all natural numbers and $S = \{(a,b,c,d) \in \mathbb{N}^4 : a^2+b^2+c^2 = d^2\}$. Find the largest positive integer m such that m divides $abcd$ for all $(a,b,c,d) \in S$.

Solution

Since $d^2 \equiv 0, 1 \pmod 4$ at most one of a, b, c is odd. Therefore 4 divides $abcd$. Also, if 3 does not divide each of a, b and c then

$$d^2 = a^2 + b^2 + c^2 \equiv 1 + 1 + 1 \equiv 0 \pmod 3.$$

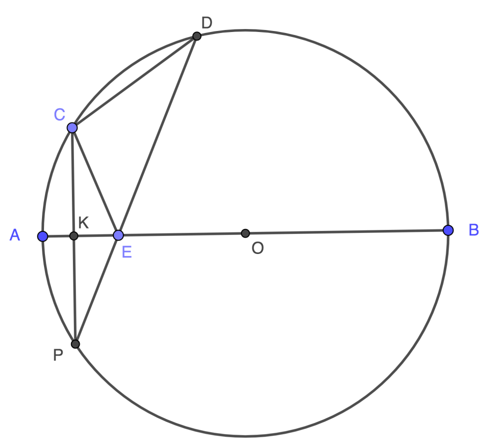
Thus 3 divides $abcd$. Therefore 12 divides $abcd$ and if m is the largest positive integer such that m divides all $abcd \in S$ then $m = 12k$ for some positive integer k. But $(1,2,2,3) \in S$ and $1.2.2.3 = 12$. Hence $k = 1$ and $m = 12$.

Remarks

The set S is infinite because $(n, n+1, n(n+1), n^2+n+1) \in S$ for every positive integer n.

2. Let ω be a semicircle with AB as the bounding diameter and let CD be a variable chord of the semicircle of constant length such that C, D lie in the interior of the arc AB. Let E be a point on AB such that CE and DE are equally inclined to the line AB. Prove that
(a) the measure of $\angle CED$ is a constant;
(b) the circumcircle of triangle CED passes through a fixed point.

Solution



Construct the circle with AB as diameter and let this circle be Ω. Draw $CK \perp AB$ with K on AB. Let CK produced meet Ω again in P. Join EP. Observe that

$$\angle DEB = \angle CEK = \angle PEK.$$

Hence $\angle PEK + \angle CEK + \angle CED = 180°$. Therefore P, E, D are collinear. This shows that

$$\angle CED = 2\angle CPD$$

is a constant. If O is the centre of Ω then we get $\angle COD = 2\angle CPD = \angle CED$. Hence the circumcircle of triangle CED passes through O which is a fixed point.

3. For any natural number n, expressed in base 10, let $s(n)$ denote the sum of all its digits. Find all natural numbers m and n such that $m < n$ and

$$(s(n))^2 = m \quad \text{and} \quad (s(m))^2 = n.$$

Solution

Let $m < n$ be such natural numbers. Let

$$m = 10^{k-1}a_{k-1} + 10^{k-2}a_{k-2} + \cdots + 10a_1 + a_0$$

be a k-digit number. Then we have

$$10^{k-1} \leq m < n = s(m)^2 = (a_{k-1} + a_{k-2} + \cdots + a_1 + a_0)^2 \leq 9^2k^2.$$

If $k \geq 5$, this is not possible. Hence $k \leq 4$.
If $k = 4$, then

$$m = 1000a_3 + 100a_2 + 10a_1 + a_0 < (a_3 + a_2 + a_1 + a_0)^2 \leq 36^2 = 1296.$$

This shows that $a_3 = 1$. In this case

$$m = 1000 + 100a_2 + 10a_1 + a_0 < (1 + a_2 + a_1 + a_0)^2 \leq 28^2 = 784,$$

which is impossible. Hence m must be a 3-digit number. Again

$$m = 100a_2 + 10a_1 + a_0 < (a_2 + a_1 + a_0)^2 \leq 27^2 = 729.$$

Hence $a_2 \leq 7$. If $a_2 = 7$, then

$$m = 700 + 10a_1 + a_0 < (7 + a_1 + a_0)^2 \leq 25^2 = 625,$$

which is not possible. Similarly, $a_2 = 6$ gives

$$m = 600 + 10a_1 + a_0 < (6 + a_1 + a_0)^2 \leq 24^2 = 576,$$

which again is impossible. If $a_2 = 5$, we obtain the maximal digital sum 23 when $a_1 = a_0 = 9$. Otherwise $s(m) \leq 22$ and

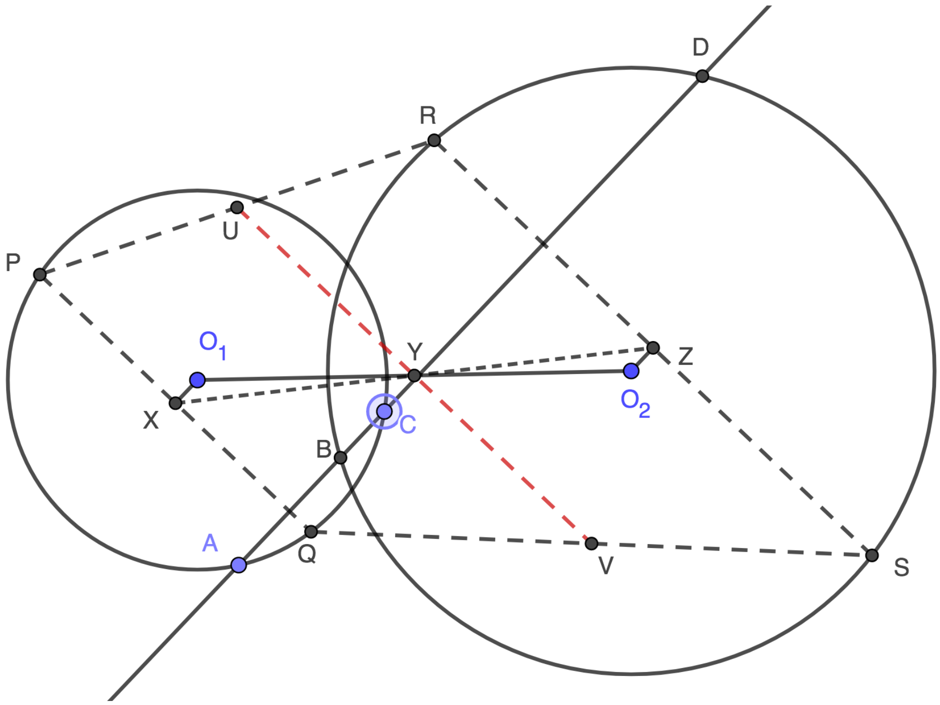
$$m = 500 + 10a_1 + a_0 < n = s(m)^2 \leq 22^2 = 484.$$

Thus we can also rule out $a_2 = 5$. Therefore $a_2 \leq 4$. This means, $m \leq 22^2$.
Now we can search which squares up to 22^2 admit an n such that $m = (s(n))^2$ and $n = (s(m))^2$. The first such square is $m = 81 = 9^2$. But in this case $n = s(m)^2 = 81$. But now $m = n$ violating $m < n$. The next square is $m = 169 = 13^2$. In this case $s(m)^2 = 16^2 = 256 = n$ and $s(n)^2 = 13^2 = 169$. Thereafter, no square satisfies this. Thus we get the pair

$$(m, n) = (169, 256).$$

4. Let Ω_1, Ω_2 be two intersecting circles with centres O_1, O_2 respectively. Let l be a line that intersects Ω_1 at points A, C and Ω_2 at points B, D such that A, B, C, D are collinear in that order. Let the perpendicular bisector of segment AB intersect Ω_1 at points P, Q; and the perpendicular bisector of segment CD intersect Ω_2 at points R, S such that P, R are on the same side of l. Prove that the midpoints of PR, QS and O_1O_2 are collinear.

Solution



Let the midpoints of segments PQ, O_1O_2, RS be denoted by X, Y, Z respectively.
We observe that B is the reflection of A in line PQ. Hence B is the orthocentre of ΔCPQ.
Hence, $O_1X = BC/2$. Similarly, $O_2Z = BC/2$.
By the S-A-S test, $\Delta XO_1Y \cong \Delta ZO_2Y$; hence $X - Y - Z$ with $XY = YZ$.
The endpoints of segments PR, XZ, QS lie on parallel lines PQ and RS, so their midpoints are collinear.

5. Let $n > k > 1$ be positive integers. Determine all positive real numbers $a_1, a_2, \dots, a_n$ which satisfy

$$\sum_{i=1}^{n} \sqrt{\frac{ka_i^k}{(k-1)a_i^k + 1}} = \sum_{i=1}^{n} a_i = n.$$

Solution 1

By A.M-G.M inequality we have

$$\frac{(k-1)a_i^k + 1}{k} \geq \left(a_i^{k(k-1)}\right)^{1/k} = a_i^{k-1}$$

which implies $\sqrt{\frac{ka_i^k}{(k-1)a_i^k + 1}} \leq \sqrt{a_i}$. Hence

$$\sum_{i=1}^{n} \sqrt{a_i} \geq \sum_{i=1}^{n} \sqrt{\frac{ka_i^k}{(k-1)a_i^k + 1}} = \sum_{i=1}^{n} a_i = n.$$

But by Cauchy-Schwarz inequality we have

$$\sum_{i=1}^{n} \sqrt{a_i} \leq \sqrt{n(\sum_{i=1}^{n} a_i)} = n.$$

Therefore

$$n \geq \sum_{i=1}^{n} \sqrt{a_i} \geq \sum_{i=1}^{n} \sqrt{\frac{ka_i^k}{(k-1)a_i^k + 1}} = \sum_{i=1}^{n} a_i = n$$

and hence equality holds everywhere which implies $a_i = 1$ for $i = 1, 2, \dots, n$.

Solution 2

Claim: For any nonnegative real number b we have $\frac{kb^k}{(k-1)b^k + 1} \leq b$.

This inequality holds iff

$$(b-1)(kb^{k-1} - \left(b^{k-1} + b^{k-2} + \cdot + 1\right)) \geq 0.$$

Observe that if $b \geq 1$ then

$$kb^{k-1} - \left(b^{k-1} + b^{k-2} + \cdot + 1\right) \geq 0$$

and if $b \leq 1$ then

$$kb^{k-1} - \left(b^{k-1} + b^{k-2} + \cdot + 1\right) \leq 0.$$

Combining these two cases we obtain

$$(b-1)(kb^{k-1} - \left(b^{k-1} + b^{k-2} + \cdot + 1\right)) \geq 0$$

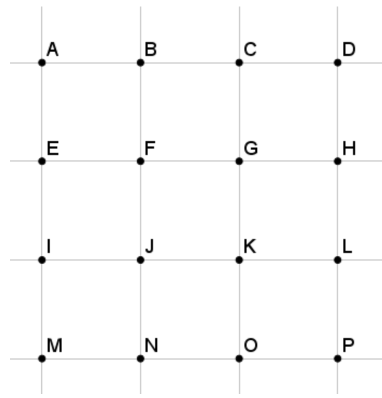
which proves our claim.
By this claim we have

$$\sqrt{\frac{ka_i^k}{(k-1)a_i^k + 1}} \leq \sqrt{a_i}$$

for $i = 1, 2, \dots, n$. The rest of the solution is the same as Solution 1.

6. Consider a set of 16 points arranged in a 4×4 square grid formation. Prove that if any 7 of these points are coloured blue, then there exists an isosceles right-angled triangle whose vertices are all blue.

Solution



Let us label the points as illustrated in the above diagram. We can consider the following cases:

Case 1: None of the central 4 points $\{F, G, J, K\}$ is colored.
We can partition the remaining 12 points into the 3 sets $\{A, D, P, M\}, \{B, H, O, I\}, \{C, L, N, E\}$. By PHP, at least 3 of the 7 colored points lie in the same set; forming a $45-45-90$ triangle (an isosceles right-angled triangle).

Case 2: At least one of the central 4 points is colored; WLOG let point F be colored.

Subcase 2.1: Points F, C are both colored.
Then, none of the points A, B, G, H, K can be colored, as each of them forms a $45-45-90$ triangle along with F, C. The remaining 9 points (out of which 5 are colored) can be partitioned into the 4 sets $\{E, I, J\}, \{D, O\}, \{L, M\}, \{N, P\}$. So by PHP, some set contains atleast 2 colored points, which form a $45-45-90$ triangle along with F.

Subcase 2.2: Point F, I are both colored. By symmetry, this is identical to subcase 2.1.

Subcase 2.3: Point F is colored, but neither C nor I is colored.
Then apart from C, F, I, the remaining 13 points (out of which 6 are colored) can be partitioned into the 5 sets $\{A, B, E\}, \{G, J, K\}, \{D, O\}, \{H, N, P\}, \{L, M\}$. So by PHP, some set contains atleast 2 colored points, which form a $45-45-90$ triangle along with F.

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