Regional Mathematical Olympiad-2023 solutions

1. Given a triangle ABC with $\angle ACB = 120^\circ$. The point L is marked on the side AB so that CL is the bisector of $\angle ACB$. The points N and K are marked on the sides AC and BC, respectively, so that $CN + CK = CL$. Prove that the triangle KLN is equilateral.

Solution 1

Let $CN = x$ and $CK = y$. Then $CL = x + y$. By using cosine rule we obtain

$$NL^2 = x^2 - x(x+y) + (x+y)^2 = x^2 + xy + y^2 = y^2 - y(x+y) + (x+y)^2 = KL^2.$$

Also

$$NK^2 = x^2 + y^2 - 2xy\cos 120^\circ = x^2 + xy + y^2.$$

Hence $NL = KL = NK$ and the triangle KLN is equilateral.

Solution 2

Taking KN as base, draw the equilateral triangle KMN such that C and M are on opposite sides of KN. Since

$$\angle KCN + \angle KMN = 180^\circ,$$

$CKMN$ is cyclic. Then

$$\angle KCM = \angle KMN = 60^\circ$$

which implies that L lies on the ray CM. Also, by application of Ptolemy's Theorem,

$$CM = CK + CN = CL$$

which implies $M \equiv L$.

2. Given a prime number p such that the number $2p$ is equal to the sum of the squares of some four consecutive positive integers. Prove that $p - 7$ is divisible by 36.

Solution

If $n > 1$ is such that

$$2p = (n-1)^2 + n^2 + (n+1)^2 + (n+2)^2 = 4n^2 + 4n + 6$$

then

$$p = 2n(n+1) + 3 > 3.$$

Observe that if $n \equiv 0 \pmod 3$ or $n \equiv 2 \pmod 3$ then $p \equiv 0 \pmod 3$ and hence can't be a prime. Therefore $n \equiv 1 \pmod 3$. Write $n = 3k + 1$ for some positive integer k. Observe that

$$p - 7 = 2(n^2 + n - 2) = 2(n-1)(n+2) = 18k(k+1) \equiv 0 \pmod{36}.$$

3. Let $f(x)$ be a polynomial with real coefficients of degree 2. Suppose that for some pairwise distinct nonzero real numbers a, b, c we have

$$f(a) = bc;\ \ f(b) = ca;\ \ f(c) = ab.$$

Determine $f(a+b+c)$ in terms of a, b, c.

Solution

Observe that

$$af(a) = bf(b) = cf(c) = abc.$$

. Let

$$P(x) = xf(x) - abc.$$

Then $P(x)$ is a polynomial of degree 3 with three distinct roots a, b, c. Therefore

$$P(x) = k(x-a)(x-b)(x-c)$$

where k is a real constant. Thus

$$xf(x) = k(x-a)(x-b)(x-c) + abc.$$

Putting $x = 0$ gives $k = 1$. Hence

$$f(x) = x^2 - (a+b+c)x + ab + bc + ca$$

and

$$f(a+b+c) = ab + bc + ca.$$

4. The set X of N four-digit numbers formed from the digits 1, 2, 3, 4, 5, 6, 7, 8 satisfies the following condition:

for any two different digits from 1, 2, 3, 4, 5, 6, 7, 8 *there exists a number in* X *which contains both of them.*

Determine the smallest possible value of N.

Solution

Let some digit, say 1, appear in exactly k numbers from the N given numbers. Hence, 1 forms at most 3 distinct pairs with the remaining 3 digits of any of these k numbers. Since the total number of all distinct pairs formed by 1 and the other 7 numbers $\{2,3,4,5,6,7,8\}$ is equal to 7, we see that $3k \geq 7$. So $k \geq 3$. Therefore, each of the digits $1,2,3,4,5,6,7,8$ must appear in at least 3 numbers. Thus, the total number of all digits in N numbers is at least $8.3 = 24$. But N numbers contain exactly $4N$ digits. Therefore, $4N \geq 24$, so $N \geq 6$.

The following example shows that there are 6 four-digit numbers satisfying the problem condition:

1234, 1567, 1268, 2357, 3468, 4578.

5. The side-lengths a, b, c of a triangle ABC are positive integers. Let

$$T_n = (a+b+c)^{2n} - (a-b+c)^{2n} - (a+b-c)^{2n} + (a-b-c)^{2n}$$

for any positive integer n. If $\dfrac{T_2}{2T_1} = 2023$ and $a > b > c$, determine all possible perimeters of the triangle ABC.

Solution

Upon simplification we obtain $T_1 = 8bc$ and $T_2 = 16bc(3a^2 + b^2 + c^2)$. Hence

$$\frac{T_2}{2T_1} = 3a^2 + b^2 + c^2.$$

Since $a > b > c$ we have $404 < a^2 < 675$ which is equivalent to $21 \le a \le 25$. Since $2023 \equiv 3 \pmod 4$, a can't be even for otherwise

$$b^2 + c^2 = 2023 - 3a^2 \equiv 3 \pmod 4$$

which is impossible because $b^2 + c^2 \equiv k \pmod 4$ where $k \le 2$. If $a = 21$ then

$$b^2 + c^2 = 2023 - 3.21^2 = 7.10^2$$

implying $b \equiv 0 \pmod 7$ and $c \equiv \pmod 7$. But then $b = 7b_1$ and $c = 7c_1$ for some positive integers b_1 and c_1 and we get

$$7(b_1^2 + c_1^2) = 10^2$$

which is absurd because 7 does not divide $10^2 = 100$. If $a = 25$ then

$$b^2 + c^2 = 48$$

implying that 3 divides $b^2 + c^2$ which is true if and only if 3 divides both b and c. Thus there exist positive integers b_2 and c_2 such that $b = 3b_2$ and $c = 3c_2$. But then we obtain

$$3(b_2^2 + c_2^2) = 16$$

which is absurd because 3 does not divide 16. When $a = 23$,

$$b^2 + c^2 = 2023 - 1587 = 436 = 20^2 + 6^2.$$

Note that $a = 23$, $b = 20$ and $c = 6$ are side-lengths of a non-degenerate triangle ABC whose perimeter is 49.

6. The diagonals AC and BD of a cyclic quadrilateral $ABCD$ meet at P. The point Q is chosen on the segment BC so that PQ is perpendicular to AC. Prove that the line joining the centres of the circumcircles of triangles APD and BQD is parallel to AD.

Solution

Choose a point T on the line QP so that $DT \perp DA$. The points A, P, D, and T are concyclic, so the center of the circle APD lies on the perpendicular bisector ℓ of DT (notice that $\ell \parallel AD$). Next, $\angle QBD = \angle PAD = \angle QTD$, so the points B, Q, D, and T are also concyclic. Therefore the centre of the circle BQD also lies on ℓ.

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