Indian National Astronomy Olympiad (INAO) – 2023

Question Paper

Roll Number: □□□□ - □□□□ - □□□□ Date: 28 January 2023
Duration: **Three Hours** Maximum Marks: 100

Please Note:

- Before starting, please ensure that you have received a copy of the question paper containing total 5 pages.
- Please write your roll number in the space provided above.
- There are total 6 questions. Maximum marks are indicated in front of each sub-question.
- For all questions, the process involved in arriving at the solution is more important than the final answer. Valid assumptions / approximations are perfectly acceptable. Please write your method clearly, explicitly stating all reasoning / assumptions / approximations.
- Use of non-programmable scientific calculators is allowed.
- **The answer-sheet must be returned to the invigilator**. You can take this question paper back with you.
- Please take note of following details about Orientation-Cum-Selection Camp (OCSC) in Astronomy:
  - Tentative Dates: 1 to 18 May 2023.
  - This camp will be held at HBCSE, Mumbai.
  - Attending the camp for the entire duration is mandatory for all participants.

Useful Constants

Mass of the Sun	$M_\odot$	$\approx$	1.989×10^{30} kg
Mass of the Earth	$M_\oplus$	$\approx$	5.972×10^{24} kg
Mass of the Moon	$M_☾$	$\approx$	7.347×10^{22} kg
Radius of the Sun	$R_\odot$	$\approx$	6.955×10^{8} m
Radius of the Earth	$R_\oplus$	$\approx$	6.371×10^{6} m
Radius of the Moon	$R_☾$	$\approx$	1.737×10^{6} m
Speed of Light	c	$\approx$	2.998×10^{8} m/s
Astronomical Unit	$a_\oplus$	$\approx$	1.496×10^{11} m
Solar Constant (at Earth)	S	$\approx$	$1366\,\mathrm{W/m^2}$
Gravitational Constant	G	$\approx$	$6.674 \times 10^{-11}\,\mathrm{N\,m^2/kg^2}$
Radius of Martian orbit	d_{Mars}	$\approx$	1.524 au

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1. For the spacecrafts travelling from one planet to another, the travel path can be designed in multiple ways. The most energy efficient of these methods is called 'Hohmann Transfer Orbit'. In this method, the spacecraft travels in an elliptical orbit with the Sun at the principal focus. Thus, its motion is governed by Kepler's Laws and hardly any fuel is spent for this journey. Step-wise process of Hohmann orbit is as follows:

- Step 1: The spacecraft is in the close proximity of Earth.
- Step 2: The spacecraft engine is fired to allow it to escape the sphere of influence of the Earth and enter the region where the Sun's gravity is the dominating force. The point in Earth's orbit at which engine is fired will also be a point in the new orbit of the spacecraft around the Sun.
- Step 3: The spacecraft travels along an elliptical orbit around the Sun and reaches the orbit of Mars.
- Step 4: The timing of this journey is such that Mars is exactly at the common point of Martian orbit and the elliptical orbit of the spacecraft at the instance when the spacecraft reaches this point.
- Step 5: The spacecraft engine is fired again so that the spacecraft is captured by the Mars' gravity and it remains in close proximity of Mars.

Points to note:

A. The point where the engine is first fired (Step 2) will be spacecraft's shortest distance from the Sun.

B. The point where the spacecraft is captured by Mars (Step 5) will be the spacecraft's farthest distance from the Sun.

C. If Step 5 is not executed, the spacecraft will complete its orbit around the Sun and reach back the point in space where Step 2 was executed.

D. Let us assume that both Earth and Mars are in circular, co-planer orbits around the Sun and the Hohmann transfer orbit is also in the same plane.

E. Assume that all the three bodies (Earth, Mars and the spacecraft) are point masses and moving in anti-clockwise direction (as viewed from the top) in their respective orbits.

F. The time durations for which engine is fired (Step 2 as well as Step 5) is very small. Hence, the thrust is almost instantaneous.

Based on this, answer the following questions:

(a) (1.5 marks) Find the orbital period of Mars in years.

Solution:

Using Kepler's law, orbital period of Mars in years:

$$P_{\mathrm{M}} = d_{\mathrm{M}}^{\frac{3}{2}}$$
$$= (1.524\,\mathrm{au})^{\frac{3}{2}}$$
$$\boxed{P_{\mathrm{M}} = 1.881\,\mathrm{yr}}$$

(b) (1.5 marks) Find the length of semi-major axis of this Hohmann transfer orbit (in au).

Solution:
The Sun-Earth distance, d_{E}, is the perihelion distance and d_{M}, i.e. the Sun-Mars distance, is the aphelion distance of the spacecraft.
Hence, semi-major axis, (a_{H}), of the Hohmann Transfer orbit is:

$$a_{\mathrm{H}} = \frac{d_{\mathrm{E}} + d_{\mathrm{M}}}{2}$$
$$\boxed{a_{\mathrm{H}} = 1.262\,\mathrm{au}}$$

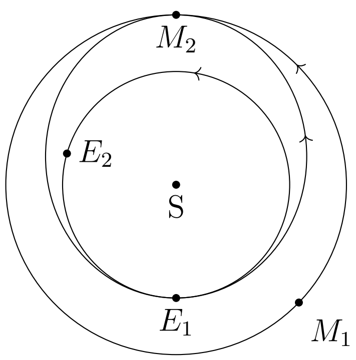
(c) (1.5 marks) Find time spent by the spacecraft between Step 2 and Step 5.

Solution:
Let (t_{H}) be the time spent between Step 2 and Step 5, by the spacecraft in the Hohmann transfer orbit.
This time will be half of the orbital period, P_{H}, of the elliptical Hohmann transfer orbit.

$$P_{\mathrm{H}} = a_{\mathrm{H}}^{3/2}$$
$$\therefore t_{\mathrm{H}} = \frac{1}{2} \times P_{\mathrm{H}}$$
$$= \frac{1}{2} \times (1.262)^{3/2}$$
$$\boxed{t_{\mathrm{H}} = 0.709\,\mathrm{yr}}$$

(d) (5 marks) Draw a diagram showing the Hohmann transfer orbit between Earth and Mars. Show positions of Earth and Mars at the instance of Step 2 (as E_1 and M_1) and at the instance of Step 5 (E_2 and M_2). Keep position E_1 directly below the Sun. Calculate angles subtended with respect to Sun - E_1 line for the other three points (M_1, E_2 and M_2).

Solution:
Hohmann Transfer Orbit between Earth and Mars can be shown as below,



From the figure above - $\boxed{\angle M_2SE_1 = 180^\circ}$

$\angle M_1SE_1$ - During spacecraft flight from Step 2 to Step 5 (0.707 years) Mars would move through an angle:

$$\begin{aligned}\theta_{2-5} &= 360^\circ \times \frac{0.709}{1.881} \\ &= 136^\circ\end{aligned}$$

$\therefore$ when spacecraft is at E_1, Mars will be at an angle $\angle E_1SM_1$ given as

$$\angle E_1SM_1 = 180^\circ - 136^\circ$$
$$\boxed{\angle E_1SM_1 = 44^\circ}$$

And the angle between $E_1 - S$ and $S - E_2$ is

$$\angle E_1SE_2 = 360^\circ \times 0.709$$
$$\boxed{\angle E_1SE_2 = 255^\circ}$$

(e) (2.5 marks) Calculate orbital velocity of Earth and Mars around Sun.

Solution:
Orbital velocity of Earth around the Sun (V_E) is:

$$\begin{aligned}V_\text{E} &= \frac{2\pi \times d_\text{E}}{P_\text{E}} \\ &= \frac{2\pi \times 1\,\text{au}}{1\,\text{yr}} \\ &= 6.28\,\text{au/yr}\end{aligned}$$
$$\boxed{V_\text{E} = 29.80\,\text{km/s}}$$

Orbital velocity of Mars around the Sun (V_M) is:

$$\begin{aligned}V_\text{M} &= \frac{2\pi \times d_\text{M}}{P_\text{M}} \\ &= \frac{2\pi \times 1.524\,\text{au}}{1.881\,\text{yr}} \\ &= 5.09\,\text{au/yr}\end{aligned}$$
$$\boxed{V_\text{M} = 24.15\,\text{km/s}}$$

(f) (8 marks) Determine if the following statements are True or False. Explain your reason in one line each.

i. The semi-minor axis of the Hohmann orbit is 1 au.

ii. The semi-major axis of the Hohmann orbit is 1.52 au.
iii. The thrust was applied in Step 2 to increase the speed of the spacecraft.
iv. The thrust was applied in Step 5 to increase the speed of the spacecraft.
v. If Step 5 is missed, we can re-capture the spacecraft in an orbit around the Earth when it again reaches its departure point in space. (see point C above).
vi. The Hohmann transfer orbit only touches (and does not intersect) the Earth's orbit.
vii. The kinetic energy of the spacecraft in Hohmann orbit is lowest just before Step 5.
viii. The spacecraft would be approaching Mars from behind in its orbit.

Solution:

i. False. Semi major axis will be greater than d_{E}.

ii. False. Semi major axis will be less than d_{M}.

iii. True. The spacecraft has to go to an orbit with $a_{\mathrm{H}} > d_{\mathrm{E}}$.

iv. True. It has to go to an orbit $d_{\mathrm{M}} > a_{\mathrm{H}}$.

v. False. The Earth would have moved from that point in its orbit.

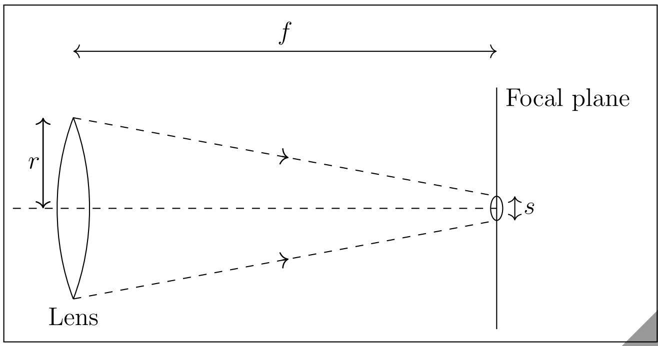
vi. True. Any point other than the perihelion point will be at a larger distance in elliptical orbit. See the figure above.

vii. True. Since the speed of the spacecraft will be lowest at the aphelion point.

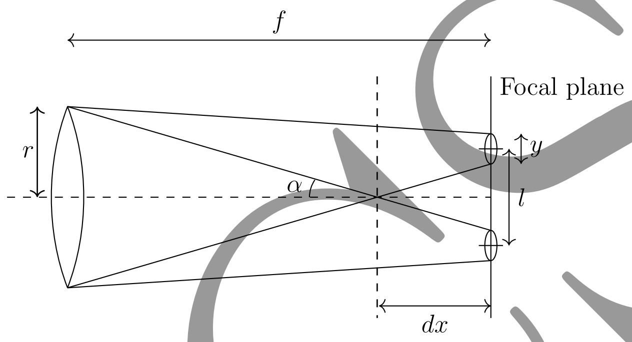
viii. False. Before step 5, velocity of the spacecraft is lower than martian orbital velocity and after step 5 it just matches the martian velocity. Hence if it was behind the Mars, it would never catch up to the Mars.

2. (10 marks) A telescope with objective lens of focal length $f = 200\,\mathrm{cm}$ is used to image a binary star system using a CCD kept at its focal plane and the image forming symmetrically at the centre (on the optic axis). In the figure, s is the image size of a point object. The image has a finite size due to diffraction. Two nearby objects are considered to be resolved when their image disks are just touching each other externally.

The radius of the objective lens is $r = 5\,\mathrm{cm}$. The limiting resolution of the lens is given to be $1''$ and the two stars are separated by an angle of $3''$. Determine the range dx allowed for the CCD to be displaced from the focal plane towards the lens such that individual stars can still be separately resolved.
(Note: This is related to the notion of "depth of field" of a camera lens.)



Solution:



From the figure above we have,

$$\tan\alpha = \frac{r}{f-dx} = \frac{l/2 - y/2}{dx}$$
$$r = \left(\frac{f}{dx} - 1\right)\left(\frac{l-y}{2}\right)$$
$$\frac{f}{dx} = \frac{2r}{l-y} + 1 = \frac{2r + l - y}{l-y}$$
$$dx = \frac{f(l-y)}{2r + l - y}$$

Typically, $r \gg l - y$ so,

$$dx = \frac{f(l-y)}{2r}$$

Now, y, the size of image of a star in the focal plane is given as,

$$y = f \times 1''$$
$$y = f/206265$$
$$y = f \times 4.85 \times 10^{-6}$$

We know, $l = 3y$

$$dx = \frac{f(3y-y)}{2r}$$
$$dx = \frac{f(2y)}{2r} = \frac{fy}{r}$$

$$dx = \frac{f \times f \times 4.85 \times 10^{-6}}{r} = \frac{f^2 \times 4.85 \times 10^{-6}}{r}$$
$$dx = \frac{(200)^2 \times 4.85 \times 10^{-6}}{5} \text{cm}$$
$$dx = 0.038 \, \text{cm}$$
$$\therefore \boxed{dx \approx 0.4 \, \text{mm}}$$

3. A cubical box, with edges of length s, is in a circular orbit of radius d around a star. The time period of revolution of the box is P. The star can be approximated as a perfect blackbody with radius R_0 ($s, R_0 \ll d$) and temperature T_0.

   In all parts of this question, at time $t = 0$, normal vector to one face of the box is exactly pointing towards the star and this face has an albedo of A (where $A \in [0, 1]$). The top face of the box is parallel to the orbital plane of the box.

   The box is in thermal equilibrium at each instance $t \geq 0$, i.e. its emissivity is exactly equal to its absorptivity. The temperature T in all parts of the box is the same at each instance.
   Note: Albedo is the fraction of incident light reflected off the surface of an object.

   (a) (7 marks) If the same face of the box is locked in towards the star at all times, find its equilibrium temperature T.

Solution:
We define angular speed as: $\omega_0 = \dfrac{2\pi}{P}$
The luminosity of the star is given by Stefan's law:

$$L_0 = 4\pi R_0^2 \sigma T_0^4$$

The flux received by the box at distance d from the star is:

$$F_{\text{rec}} = \frac{L_0}{4\pi d^2}$$

If at all times, a single face of the cube (with projected area s^2 & albedo A) is exposed to the incoming stellar radiation, then the amount of power absorbed by the box at any given time is:

$$\begin{aligned} P_{\text{abs}}(t) &= (1 - A) \times F_{\text{rec}} \times s^2 \\ &= (1 - A) \times \frac{L_0}{4\pi d^2} \times s^2 \\ &= (1 - A) \frac{\cancel{4\pi} R_0^2 \sigma T_0^4}{\cancel{4\pi} d^2} s^2 \\ P_{\text{abs}}(t) &= (1 - A) \frac{s^2}{d^2} R_0^2 \sigma T_0^4 \end{aligned}$$

Now, because the box (total surface area of $6s^2$) is always in thermal equilibrium with the absorbed radiation, hence, by Stefan's law:

$$6\cancel{s^2}\cancel{\sigma}T^4 = (1-A)\frac{\cancel{s^2}}{d^2}\,R_0^2\cancel{\sigma}T_0^4$$
$$T^4 = (1-A)\frac{R_0^2}{6d^2}\,T_0^4$$
$$\boxed{T = T_0\left[(1-A)\frac{R_0^2}{6d^2}\right]^{1/4}}$$

(b) (6 marks) Let all 6 faces of the box have albedo A. If the box is revolving around the star, but not rotating about itself, derive the expression for the equilibrium temperature T as a function of time t.

Solution:
At time t, the total projected area Φ onto which the stellar radiation falls is:

$$\Phi = s^2\,[\,|\cos(\omega_0 t)| + |\sin(\omega_0 t)|\,]$$

where, the cosine term comes from the side that was initially facing the star and the side opposite to it (the absolute value compensates for one of them receiving light when the other doesn't) and the sine term similarly, from the remaining 2 sides on which radiation can fall.
Hence, the net power absorbed by the box at time t is:

$$P_{\text{abs}}(t) = (1-A)F_{\text{rec}} \times \Phi$$
$$P_{\text{abs}}(t) = (1-A)\frac{R_0^2\,\sigma T_0^4}{d^2} \times s^2\,[\,|\cos(\omega_0 t)| + |\sin(\omega_0 t)|\,]$$

Therefore, Stefan's law gives:

$$6\cancel{s^2}\cancel{\sigma}T(t)^4 = (1-A)\frac{R_0^2\cancel{\sigma}T_0^4}{d^2}\cancel{s^2}\,[\,|\cos(\omega_0 t)| + |\sin(\omega_0 t)|\,]$$
$$\boxed{T(t) = T_0\left[\frac{R_0^2}{6d^2}\right]^{1/4}[\,(1-A)\,|\cos(\omega_0 t)| + (1-A)\,|\sin(\omega_0 t)|\,]^{1/4}}$$

(c) (3 marks) Now consider the case, where the face of the cube which is towards the star at $t = 0$ and the face opposite to it have albedo A and all other faces of the cube have albedo of $1-A$. Observe the expressions obtained in part b and write the final expression for temperature in this case.

Solution:

The expression obtained in part b is -

$$T(t) = T_0 \left[\frac{R_0^2}{6d^2}\right]^{1/4} \left[\ (1-A)\,|\cos(\omega_0 t)| + (1-A)\,|\sin(\omega_0 t)|\ \right]^{1/4}$$

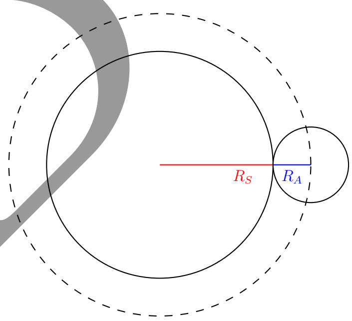
Let us consider the extreme case, at $t = 0$, the face towards the star (and that opposite to it) which has albedo A (and hence absorption fraction of $1 - A$) should contribute to the temperature. While the contribution of the other faces will be zero. Hence, the cosine term, which survives at $t = 0$ will have a multiplying factor of $1 - A$ and thus the multiplying factor for the sine term will be A.
Thus we get,

$$\boxed{T(t) = T_0 \left[\frac{R_0^2}{6d^2}\right]^{1/4} \quad [(1-A)\,|\cos(\omega_0 t)| + A\,|\sin(\omega_0 t)|]^{1/4}}$$

4. Indian Space Research Organisation (ISRO) decided to send a spaceship to the planet Jupiter. The spaceship needs to pass through the asteroid belt between Mars and Jupiter which lies between 2.0 au and 3.0 au. The asteroid belt has a thickness (total height perpendicular to the plane of the solar system) of about 0.10 au and there are about 10^{14} asteroids of 1.0 m radius (assumed to be spherical) in it. The spaceship is spherical and is 3.0 m in radius.

   (a) (2 marks) Calculate the cross-section area within which the centre of an asteroid needs to come from the centre of the spacecraft to collide with it.

Solution:
Cross section area diagram.

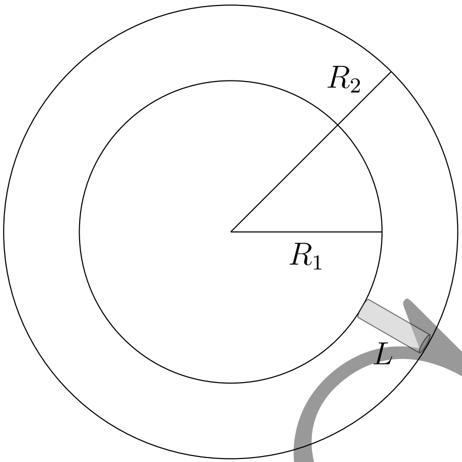


The cross-section area (σ) can be calculated as follows:

$$\sigma = \pi \times (R_A + R_B)^2$$
$$= \pi \times (3.0 + 1.0)^2$$
$$\boxed{\sigma \approx 50\,\mathrm{m}^2}$$

   (b) (8 marks) Assume that the asteroid belt is completely static and the spaceship is launched on a random day of the year to travel radially away from Sun, but within the plane of solar system. Calculate the probability (P_0) that the spaceship will escape the asteroid belt unharmed.

Solution:



To find cylindrical volume traversed by the spacecraft inside the asteroid belt.
R_1 : The inner edge of the asteroid belt = 2 au
R_2 : The outer edge of the asteroid belt = 3 au
H : Height / thickness of the asteroid belt = 0.1 au
L : Path length of the spacecraft inside the belt = 1 au
Cylindrical volume along the path length will be:

$$V_{\text{path-length}} = \sigma \times L$$
$$V_{\text{belt}} = \pi \times (R_2^2 - R_1^2) \times H$$

If $\bar{n}$ is the average number density of asteroids (in unit volume) and N be the total number of asteroids.

$$\bar{n} = \frac{N}{\pi \times (R_2^2 - R_1^2) \times H}$$
$$= \frac{10^{14}}{\pi \times (1.12 \times 10^{23}) \times 1.496 \times 10^{10}}$$
$$\bar{n} = 1.9 \times 10^{-20}/\text{m}^3$$

Hence, the total number of collisions, (T_{col}), can be given as,

$$T_{\text{col}} = \bar{n}\sigma L$$
$$= 1.9 \times 10^{-20} \times 50 \times 1.496 \times 10^{11}$$
$$T_{\text{col}} = 1.4 \times 10^{-7}$$

The probability that the spacecraft will escape unharmed is

$$\boxed{P_0 = 1 - 1.4 \times 10^{-7}}$$

(c) Qualitatively compare the probability in the each of the following cases with P_0:

i. (2 marks) The probability (P_{A}) when the spacecraft's speed through the asteroid belt is doubled.

ii. (1 mark) The probability (P_B) when the spacecraft is not moving radially but is in an elliptical orbit with aphelion point at Jupiter (5 au). Here we consider probability only for a single crossing.

iii. (2 marks) The probability (P_C) when the spacecraft is not moving radially but is in an elliptical orbit with aphelion point at the centre of the asteroid belt (2.5 au).

Solution:

i. Since the equation for the probability of collisions is independent of velocity of the spacecraft, therefore, $P_0 = P_A$.

ii. When the orbit is elliptical the path covered will be more than that in the case of radial path (which is a straight line path). Hence, $P_0 > P_B$.

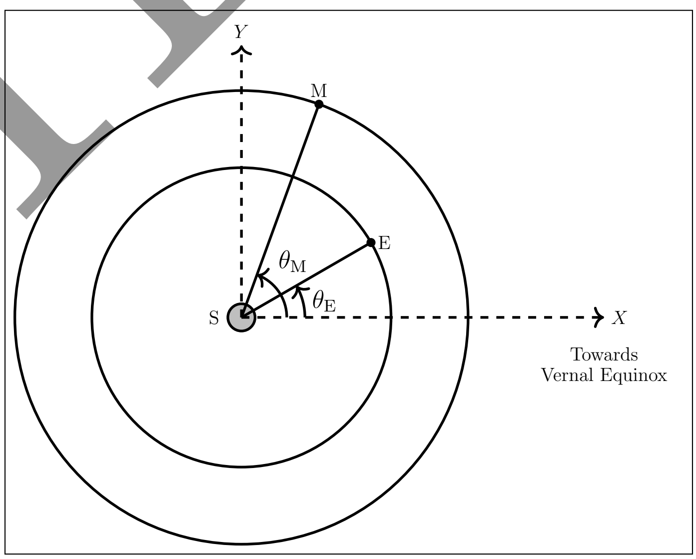
iii. In this case the spacecraft travels a longer path through the asteroid belt, hence, $P_0 > P_B > P_C$.

5. (22 marks) Assume that the Earth and Mars revolve around the Sun in coplanar circular orbits of radius r_E and r_M respectively. In the figure below, θ_E and θ_M are the angles made by the radius vectors of Earth and Mars, measured at the Sun with respect to the X-axis (which points in the direction of Vernal Equinox - a reference point in space).
On 21 September, θ_E was 0° and θ_M was 42.3°.

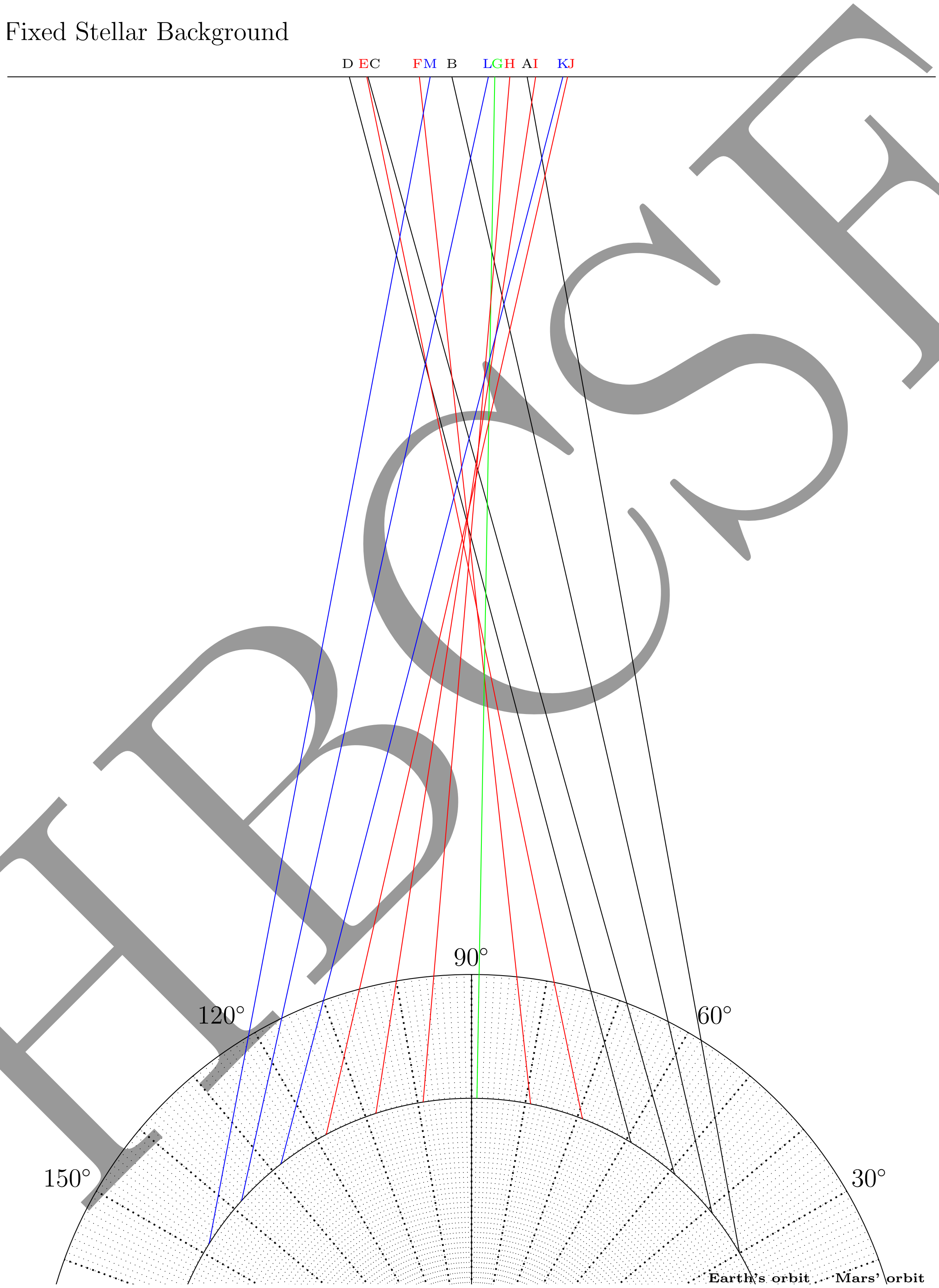
In the answersheet you are given a table to note down your observations. Calculate the values of θ_E and θ_M for the dates mentioned in the table and write them in the respective columns.

The answersheet also has a particular section of the orbits of Mars and Earth. The dotted lines are markings for angles in the orbits at an interval of 1°. The central black line is when θ_E or θ_M will be 90°. You can also see the distant stellar background near one edge of the page.

For the dates given in the table mark the projected positions of Mars on the stellar background. Also write the corresponding observation numbers near the marked projected positions. Now describe the motion of Mars as seen by an Earth based observer.



Solution:



6. Saee is an observer at North Pole during the month of December. On the grid given in your

answersheet:

(a) (1 mark) Mark the directions.

(b) (1 mark) Mark the pole star (Polaris).

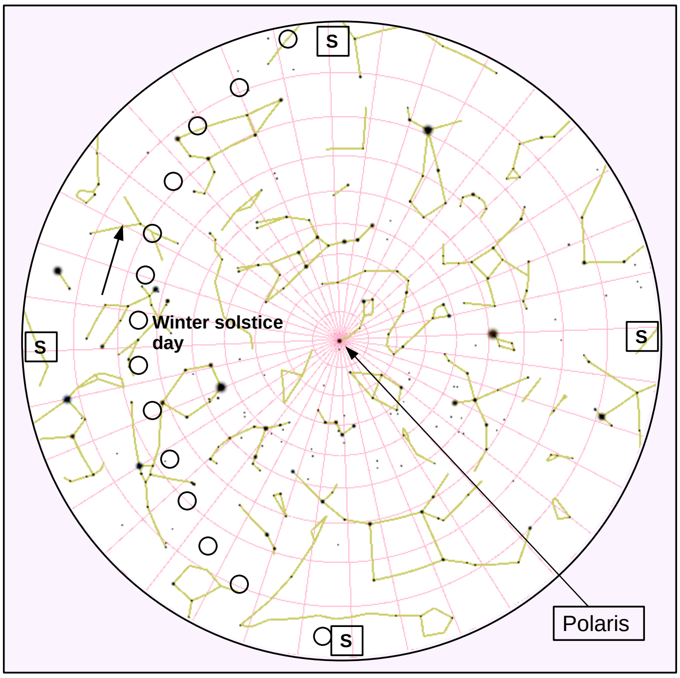
(c) (6 marks) Approximately, for how many days will the Moon be visible in one lunar month assuming clear skies? If you were doing an exact calculation, which secondary effects you will have to consider? List any three effects that may affect your answer by an hour or more.

(d) (6 marks) In this month the full Moon was seen on winter solstice day in the Gemini constellation. Saee noted the Moon's position every day at 7 pm during an entire lunar month. Mark the approximate positions of the Moon as seen by her. For each observation label the day number starting with day of first observation.

(e) (3 marks) Aadarsh, who is also camping at the North pole with Saee, noted the Moon's position for 24 hours period of the 6th day after full moon. Describe what he will observe.

Solution:

(a)



(b) See the grid

(c) 14.5 or 15 Days. Following are some of the secondary effects which can change the answer by about an hour -

- The Moon is not a point object but has a finite angular size of around half a degree as seen from Earth.

- The orbit of Moon is inclined to the ecliptic by about 5°.
- Refraction due to the atmosphere of Earth.
- Change in the position of Earth around Sun.
- Precession of nodes of lunar orbit.
- Eccentricity of lunar orbit.

(d) Everyday the Moon changes its position as shown in the grid. The direction of motion is clockwise.

(e) The Moon remains nearly in the same position with respect to the constellations. It will complete a full circle around zenith in an anticlockwise manner during the 24 hour period. After 24 hours the Moons position on the ecliptic would have shifted by 13°. Thus the two ends of the circle won't meet.