25th Indian National Mathematical Olympiad, 2010

Time: 4 hours **January 17, 2010**

- **Calculators (in any form)/Mathematical tables/Slide rules are not allowed.**
- **Answer to each question should start on a new page. Clearly indicate the question number.**
- **Draw neat figures using pencil only. Draw them on the left hand side of the page for easy reference.**

1. Let ABC be a triangle with circum-circle Γ. Let M be a point in the interior of triangle ABC which is also on the bisector of $\angle A$. Let AM, BM, CM meet Γ in A_1, B_1, C_1 respectively. Suppose P is the point of intersection of A_1C_1 with AB; and Q is the point of intersection of A_1B_1 with AC. Prove that PQ is parallel to BC.

2. Find all natural numbers $n > 1$ such that n^2 **does not** divide $(n-2)!$.

3. Find all non-zero real numbers x, y, z which satisfy the system of equations:
$$(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2) = xyz,$$
$$(x^4+x^2y^2+y^4)(y^4+y^2z^2+z^4)(z^4+z^2x^2+x^4) = x^3y^3z^3.$$

4. How many 6-tuples $(a_1, a_2, a_3, a_4, a_5, a_6)$ are there such that each of $a_1, a_2, a_3, a_4, a_5, a_6$ is from the set $\{1, 2, 3, 4\}$ and the six expressions
$$a_j^2 - a_ja_{j+1} + a_{j+1}^2$$
for $j = 1, 2, 3, 4, 5, 6$ (where a_7 is to be taken as a_1) are all equal to one another?

5. Let ABC be an acute-angled triangle with altitude AK. Let H be its ortho-centre and O be its circum-centre. Suppose KOH is an acute-angled triangle and P its circum-centre. Let Q be the reflection of P in the line HO. Show that Q lies on the line joining the mid-points of AB and AC.

6. Define a sequence $\langle a_n \rangle_{n\geq 0}$ by $a_0 = 0$, $a_1 = 1$ and
$$a_n = 2a_{n-1} + a_{n-2},$$
for $n \geq 2$.

   (a) For every $m > 0$ and $0 \leq j \leq m$, prove that $2a_m$ divides $a_{m+j} + (-1)^j a_{m-j}$.

   (b) Suppose 2^k divides n for some natural numbers n and k. Prove that 2^k divides a_n.

INMO-2010 Problems and Solutions

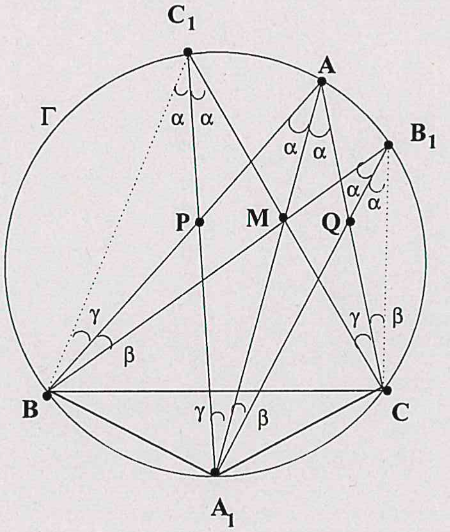
1. Let ABC be a triangle with circum-circle Γ. Let M be a point in the interior of triangle ABC which is also on the bisector of $\angle A$. Let AM, BM, CM meet Γ in A_1, B_1, C_1 respectively. Suppose P is the point of intersection of A_1C_1 with AB; and Q is the point of intersection of A_1B_1 with AC. Prove that PQ is parallel to BC.

Solution: Let $A = 2\alpha$. Then $\angle A_1AC = \angle BAA_1 = \alpha$. Thus

$$\angle A_1B_1C = \alpha = \angle BB_1A_1 = \angle A_1C_1C = \angle BC_1A_1.$$

We also have $\angle B_1CQ = \angle AA_1B_1 = \beta$, say. It follows that triangles MA_1B_1 and QCB_1 are similar and hence

$$\frac{QC}{MA_1} = \frac{B_1C}{B_1A_1}.$$



Similarly, triangles ACM and C_1A_1M are similar and we get

$$\frac{AC}{AM} = \frac{C_1A_1}{C_1M}.$$

Using the point P, we get similar ratios:

$$\frac{PB}{MA_1} = \frac{C_1B}{A_1C_1}, \quad \frac{AB}{AM} = \frac{A_1B_1}{MB_1}.$$

Thus,

$$\frac{QC}{PB} = \frac{A_1C_1 \cdot B_1C}{C_1B \cdot B_1A_1},$$

and

$$\begin{aligned} \frac{AC}{AB} &= \frac{MB_1 \cdot C_1A_1}{A_1B_1 \cdot C_1M} \\ &= \frac{MB_1}{C_1M}\frac{C_1A_1}{A_1B_1} = \frac{MB_1}{C_1M}\frac{C_1B \cdot QC}{PB \cdot B_1C}. \end{aligned}$$

However, triangles C_1BM and B_1CM are similar, which gives

$$\frac{B_1C}{C_1B} = \frac{MB_1}{MC_1}.$$

Putting this in the last expression, we get

$$\frac{AC}{AB} = \frac{QC}{PB}.$$

We conclude that PQ is parallel to BC.

2. Find all natural numbers $n > 1$ such that n^2 **does not** divide $(n-2)!$.

Solution: Suppose $n = pqr$, where $p < q$ are primes and $r > 1$. Then $p \geq 2$, $q \geq 3$ and $r \geq 2$, not necessarily a prime. Thus we have

$$\begin{aligned} n-2 &\geq n-p = pqr - p \geq 5p > p, \\ n-2 &\geq n-q = q(pr-1) \geq 3q > q, \\ n-2 &\geq n-pr = pr(q-1) \geq 2pr > pr, \\ n-2 &\geq n-qr = qr(p-1) \geq qr. \end{aligned}$$

Observe that p, q, pr, qr are all distinct. Hence their product divides $(n-2)!$. Thus $n^2 = p^2q^2r^2$ divides $(n-2)!$ in this case. We conclude that either $n = pq$ where p, q are distinct primes or $n = p^k$ for some prime p.

Case 1. Suppose $n = pq$ for some primes p, q, where $2 < p < q$. Then $p \geq 3$ and $q \geq 5$. In this case

$$\begin{aligned} n-2 &> n-p = p(q-1) \geq 4p, \\ n-2 &> n-q = q(p-1) \geq 2q. \end{aligned}$$

Thus $p, q, 2p, 2q$ are all distinct numbers in the set $\{1, 2, 3, \ldots, n-2\}$. We see that $n^2 = p^2q^2$ divides $(n-2)!$. We conclude that $n = 2q$ for some prime $q \geq 3$. Note that $n - 2 = 2q - 2 < 2q$ in this case so that n^2 does not divide $(n-2)!$.

Case 2. Suppose $n = p^k$ for some prime p. We observe that $p, 2p, 3p, \ldots (p^{k-1}-1)p$ all lie in the set $\{1, 2, 3, \ldots, n-2\}$. If $p^{k-1} - 1 \geq 2k$, then there are at least $2k$ multiples of p in the set $\{1, 2, 3, \ldots, n-2\}$. Hence $n^2 = p^{2k}$ divides $(n-2)!$. Thus $p^{k-1} - 1 < 2k$. If $k \geq 5$, then $p^{k-1} - 1 \geq 2^{k-1} - 1 \geq 2k$, which may be proved by an easy induction. Hence $k \leq 4$. If $k = 1$, we get $n = p$, a prime. If $k = 2$, then $p - 1 < 4$ so that $p = 2$ or 3; we get $n = 2^2 = 4$ or $n = 3^2 = 9$. For $k = 3$, we have $p^2 - 1 < 6$ giving $p = 2$; $n = 2^3 = 8$ in this case. Finally, $k = 4$ gives $p^3 - 1 < 8$. Again $p = 2$ and $n = 2^4 = 16$. However $n^2 = 2^8$ divides $14!$ and hence is not a solution.

Thus $n = p, 2p$ for some prime p or $n = 8, 9$. It is easy to verify that these satisfy the conditions of the problem.

3. Find all non-zero real numbers x, y, z which satisfy the system of equations:

$$\begin{aligned} (x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2) &= xyz, \\ (x^4+x^2y^2+y^4)(y^4+y^2z^2+z^4)(z^4+z^2x^2+x^4) &= x^3y^3z^3. \end{aligned}$$

Solution: Since $xyz \neq 0$, We can divide the second relation by the first. Observe that

$$x^4 + x^2y^2 + y^4 = (x^2+xy+y^2)(x^2-xy+y^2),$$

holds for any x, y. Thus we get

$$(x^2-xy+y^2)(y^2-yz+z^2)(z^2-zx+x^2) = x^2y^2z^2.$$

However, for any real numbers x, y, we have

$$x^2 - xy + y^2 \geq |xy|.$$

Since $x^2y^2z^2 = |xy|\,|yz|\,|zx|$, we get

$$|xy|\,|yz|\,|zx| = (x^2 - xy + y^2)(y^2 - yz + z^2)(z^2 - zx + x^2) \geq |xy|\,|yz|\,|zx|.$$

This is possible only if

$$x^2 - xy + y^2 = |xy|, \quad y^2 - yz + z^2 = |yz|, \quad z^2 - zx + x^2 = |zx|,$$

hold simultaneously. However $|xy| = \pm xy$. If $x^2 - xy + y^2 = -xy$, then $x^2 + y^2 = 0$ giving $x = y = 0$. Since we are looking for nonzero x, y, z, we conclude that $x^2 - xy + y^2 = xy$ which is same as $x = y$. Using the other two relations, we also get $y = z$ and $z = x$. The first equation now gives $27x^6 = x^3$. This gives $x^3 = 1/27$(since $x \neq 0$), or $x = 1/3$. We thus have $x = y = z = 1/3$. These also satisfy the second relation, as may be verified.

4. How many 6-tuples $(a_1, a_2, a_3, a_4, a_5, a_6)$ are there such that each of $a_1, a_2, a_3, a_4, a_5, a_6$ is from the set $\{1, 2, 3, 4\}$ and the six expressions

$$a_j^2 - a_j a_{j+1} + a_{j+1}^2$$

for $j = 1, 2, 3, 4, 5, 6$(where a_7 is to be taken as a_1) are all equal to one another?

Solution: Without loss of generality, we may assume that a_1 is the largest among $a_1, a_2, a_3, a_4, a_5, a_6$. Consider the relation

$$a_1^2 - a_1a_2 + a_2^2 = a_2^2 - a_2a_3 + a_3^2.$$

This leads to

$$(a_1 - a_3)(a_1 + a_3 - a_2) = 0.$$

Observe that $a_1 \geq a_2$ and $a_3 > 0$ together imply that the second factor on the left side is positive. Thus $a_1 = a_3 = \max\{a_1, a_2, a_3, a_4, a_5, a_6\}$. Using this and the relation

$$a_3^2 - a_3a_4 + a_4^2 = a_4^2 - a_4a_5 + a_5^2,$$

we conclude that $a_3 = a_5$ as above. Thus we have

$$a_1 = a_3 = a_5 = \max\{a_1, a_2, a_3, a_4, a_5, a_6\}.$$

Let us consider the other relations. Using

$$a_2^2 - a_2a_3 + a_3^2 = a_3^2 - a_3a_4 + a_4^2,$$

we get $a_2 = a_4$ or $a_2 + a_4 = a_3 = a_1$. Similarly, two more relations give either $a_4 = a_6$ or $a_4 + a_6 = a_5 = a_1$; and either $a_6 = a_2$ or $a_6 + a_2 = a_1$. Let us give values to a_1 and count the number of six-tuples in each case.

(A) Suppose $a_1 = 1$. In this case all a_j's are equal and we get only one six-tuple $(1, 1, 1, 1, 1, 1)$.

(B) If $a_1 = 2$, we have $a_3 = a_5 = 2$. We observe that $a_2 = a_4 = a_6 = 1$ or $a_2 = a_4 = a_6 = 2$. We get two more six-tuples: $(2, 1, 2, 1, 2, 1)$, $(2, 2, 2, 2, 2, 2)$.

(C) Taking $a_1 = 3$, we see that $a_3 = a_5 = 3$. In this case we get nine possibilities for (a_2, a_4, a_6);

$$(1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 1, 2), (1, 2, 1), (2, 1, 1), (1, 2, 2), (2, 1, 2), (2, 2, 1).$$

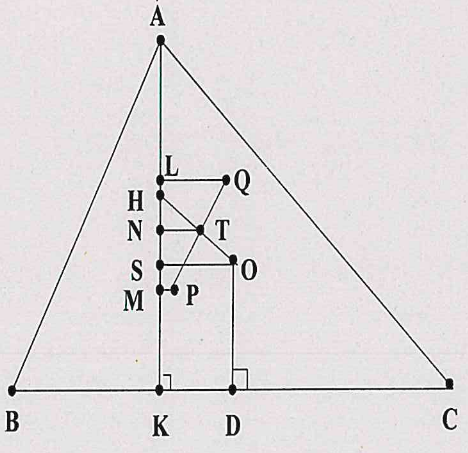
(D) In the case $a_1 = 4$, we have $a_3 = a_5 = 4$ and

$$(a_2, a_4, a_6) = (2,2,2), (4,4,4), (1,1,1), (3,3,3),$$
$$(1,1,3), (1,3,1), (3,1,1), (1,3,3), (3,1,3), (3,3,1).$$

Thus we get $1 + 2 + 9 + 10 = 22$ solutions. Since (a_1, a_3, a_5) and (a_2, a_4, a_6) may be interchanged, we get 22 more six-tuples. However there are 4 common among these, namely, $(1,1,1,1,1,1)$, $(2,2,2,2,2,2)$, $(3,3,3,3,3,3)$ and $(4,4,4,4,4,4)$. Hence the total number of six-tuples is $22 + 22 - 4 = 40$.

5. Let ABC be an acute-angled triangle with altitude AK. Let H be its ortho-centre and O be its circum-centre. Suppose KOH is an acute-angled triangle and P its circum-centre. Let Q be the reflection of P in the line HO. Show that Q lies on the line joining the mid-points of AB and AC.

Solution: Let D be the mid-point of BC; M that of HK; and T that of OH. Then PM is perpendicular to HK and PT is perpendicular to OH. Since Q is the reflection of P in HO, we observe that P, T, Q are collinear, and $PT = TQ$. Let QL, TN and OS be the perpendiculars drawn respectively from Q, T and O on to the altitude AK.(See the figure.)



We have $LN = NM$, since T is the mid-point of QP; $HN = NS$, since T is the mid-point of OH; and $HM = MK$, as P is the circum-centre of KHO. We obtain

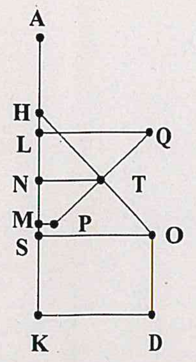
$$LH + HN = LN = NM = NS + SM,$$

which gives $LH = SM$. We know that $AH = 2OD$. Thus

$$AL = AH - LH = 2OD - LH = 2SK - SM = SK + (SK - SM) = SK + MK$$
$$= SK + HM = SK + HS + SM = SK + HS + LH = SK + LS = LK.$$

This shows that L is the mid-point of AK and hence lies on the line joining the midpoints of AB and AC. We observe that the line joining the mid-points of AB and AC is also perpendicular to AK. Since QL is perpendicular to AK, we conclude that Q also lies on the line joining the mid-points of AB and AC.

Remark: It may happen that H is above L as in the adjoining figure, but the result remains true here as well. We have $HN = NS$, $LN = NM$, and $HM = MK$ as earlier. Thus $HN = HL + LN$ and $NS = SM + NM$ give $HL = SM$. Now $AL = AH + HL = 2OD + SM = 2SK + SM = SK + (SK + SM) = SK + MK = SK + HM = SK + HL + LM = SK + SM + LM = LK$. The conclusion that Q lies on the line joining the mid-points of AB and AC follows as earlier.



6. Define a sequence $\langle a_n \rangle_{n \geq 0}$ by $a_0 = 0$, $a_1 = 1$ and

$$a_n = 2a_{n-1} + a_{n-2},$$

for $n \geq 2$.

(a) For every $m > 0$ and $0 \leq j \leq m$, prove that $2a_m$ divides $a_{m+j} + (-1)^j a_{m-j}$.

(b) Suppose 2^k divides n for some natural numbers n and k. Prove that 2^k divides a_n.

Solution:

(a) Consider $f(j) = a_{m+j} + (-1)^j a_{m-j}$, $0 \leq j \leq m$, where m is a natural number. We observe that $f(0) = 2a_m$ is divisible by $2a_m$. Similarly,

$$f(1) = a_{m+1} - a_{m-1} = 2a_m$$

is also divisible by $2a_m$. Assume that $2a_m$ divides $f(j)$ for all $0 \leq j < l$, where $l \leq m$. We prove that $2a_m$ divides $f(l)$. Observe

$$\begin{aligned} f(l-1) &= a_{m+l-1} + (-1)^{l-1} a_{m-l+1}, \\ f(l-2) &= a_{m+l-2} + (-1)^{l-2} a_{m-l+2}. \end{aligned}$$

Thus we have

$$\begin{aligned} a_{m+l} &= 2a_{m+l-1} + a_{m+l-2} \\ &= 2f(l-1) - 2(-1)^{l-1} a_{m-l+1} + f(l-2) - (-1)^{l-2} a_{m-l+2} \\ &= 2f(l-1) + f(l-2) + (-1)^{l-1}\left(a_{m-l+2} - 2a_{m-l+1}\right) \\ &= 2f(l-1) + f(l-2) + (-1)^{l-1} a_{m-l}. \end{aligned}$$

This gives

$$f(l) = 2f(l-1) + f(l-2).$$

By induction hypothesis $2a_m$ divides $f(l-1)$ and $f(l-2)$. Hence $2a_m$ divides $f(l)$. We conclude that $2a_m$ divides $f(j)$ for $0 \leq j \leq m$.

(b) We see that $f(m) = a_{2m}$. Hence $2a_m$ divides a_{2m} for all natural numbers m. Let $n = 2^k l$ for some $l \geq 1$. Taking $m = 2^{k-1} l$, we see that $2a_m$ divides a_n. Using an easy induction , we conclude that $2^k a_l$ divides a_n. In particular 2^k divides a_n.
