IOQM 2021-22 Part A

1. Three parallel lines L_1, L_2, L_3 are drawn in the plane such that the perpendicular distance between L_1 and L_2 is 3 and the perpendicular distance between L_2 and L_3 is also 3. A square $ABCD$ is constructed such that A lies on L_1, B lies on L_3 and C lies on L_2. Find the area of the square.

2. Ria writes down the numbers $1, 2, \ldots, 101$ in red and blue pens. The largest blue number is equal to the number of numbers written in blue and the smallest red number is equal to half the number of numbers written in red. How many numbers did Ria write with red pen?

3. Consider the set $\mathcal{T}$ of all triangles whose sides are distinct prime numbers which are also in arithmetic progression. Let $\triangle \in \mathcal{T}$ be the triangle with the least perimeter. If a° is the largest angle of $\triangle$ and if L is its perimeter, determine the value of $\frac{a}{L}$.

4. Consider the set of all 6-digit numbers consisting of only 3 digits, a, b, c, where a, b, c are distinct. Suppose the sum of all these numbers is 593999406. What is the largest remainder when the three digit number abc is divided by 100?

5. In parallelogram $ABCD$ the longer side is twice the shorter side. Let $XYZW$ be the quadrilateral formed by the internal bisectors of the angles of $ABCD$. If the area of $XYZW$ is 10, find the area of $ABCD$.

6. Let x, y, z be positive real numbers such that $x^2 + y^2 = 49$, $y^2 + yz + z^2 = 36$ and $x^2 + \sqrt{3}xz + z^2 = 25$. If the value of $2xy + \sqrt{3}yz + zx$ can be written as $p\sqrt{q}$ where p, q are integers and q is not divisible by square of any prime number, find $p + q$.

7. Find the number of maps $f : \{1, 2, 3\} \longrightarrow \{1, 2, 3, 4, 5\}$ such that $f(i) \leq f(j)$ whenever $i < j$.

8. For any real number t, let $\lfloor t \rfloor$ denote the largest integer $\leq t$. Suppose that N is the greatest integer such that

$$\left\lfloor \sqrt{\left\lfloor \sqrt{\left\lfloor \sqrt{N} \right\rfloor} \right\rfloor} \right\rfloor = 4$$

Find the sum of digits of N.

9. Let $P_0 = (3, 1)$ and define $P_{n+1} = (x_n, y_n)$ for $n \geq 0$ by

$$x_{n+1} = -\frac{3x_n - y_n}{2}, \qquad y_{n+1} = -\frac{x_n + y_n}{2}$$

Find the area of the quadrilateral formed by the points $P_{96}, P_{97}, P_{98}, P_{99}$.

10. Suppose that P is the polynomial of least degree with integer coefficients such that $P\left(\sqrt{7} + \sqrt{5}\right) = 2\left(\sqrt{7} - \sqrt{5}\right)$. Find $P(2)$.

11. In how many ways can four married couples sit in a merry-go-round with identical seats such that men and women occupy alternate seats and no husband seats next to his wife?

12. A 12×12 board is divided into 144 unit squares by drawing lines parallel to the sides. Two rooks placed on two unit squares are said to be non attacking if they are not in the same column or same row. Find the least number N such that if N rooks are placed on the unit squares, one rook per square, we can always find 7 rooks such that no two are attacking each other.