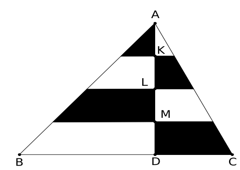
Solutions to RMO-2014 problems

1. Let ABC be a triangle and let AD be the perpendicular from A on to BC. Let K, L, M be points on AD such that $AK = KL = LM = MD$. Draw lines parallel to BC through K, L, M. If the sum of the areas of the shaded regions is equal to the sum of the areas of the unshaded regions, prove that $BD = DC$.



Solution: let $BD = 4x$, $DC = 4y$ and $AD = 4h$. Then the sum of the areas of the shaded regions is

$$\frac{1}{2}h\big(x + (y + 2y) + (2x + 3x) + (3y + 4y)\big) = \frac{h(6x + 10y)}{2}.$$

The sum of the areas of the unshaded regions is

$$\frac{1}{2}h\big(y + (x + 2x) + (2y + 3y) + (3x + 4x)\big) = \frac{h(10x + 6y)}{2}.$$

Therefore the given condition implies that

$$6x + 10y = 10x + 6y.$$

This gives $x = y$. Hence $BD = DC$.

2. Let $a_1, a_2, \ldots, a_{2n}$ be an arithmetic progression of positive real numbers with common difference d. Let
(i) $a_1^2 + a_3^2 + \cdots + a_{2n-1}^2 = x$, (ii) $a_2^2 + a_4^2 + \cdots + a_{2n}^2 = y$, and
(iii) $a_n + a_{n+1} = z$.
Express d in terms of x, y, z, n.

Solution: Observe that

$$y - x = (a_2^2 - a_1^2) + (a_4^2 - a_3^2) + \cdots + (a_{2n}^2 - a_{2n-1}^2).$$

The general difference is

$$a_{2k}^2 - a_{2k-1}^2 = \Big(a_{2k} + a_{2k-1}\Big)d = \Big(2a_1 + \big((2k-1) + (2k-2)\big)d\Big)d.$$

Therefore

$$y - x = \big(2na_1 + (1 + 2 + 3 + \cdots (2n-1))d\big)d = nd\big(2a_1 + (2n-1)d\big).$$

We also observe that

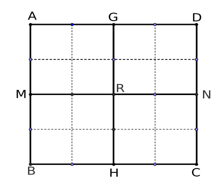
$$z = a_n + a_{n+1} = 2a_1 + (2n-1)d.$$

It follows that $y - x = ndz$. Hence $d = (y - x)/nz$.

3. Suppose for some positive integers r and s, the number 2^r is obtained by permuting the digits of the number 2^s in decimal expansion. Prove that $r = s$.

   Solution: Suppose $s \leq r$. If $s < r$ then $2^s < 2^r$. Since the number of digits in 2^s and 2^r are the same, we have $2^r < 10 \times 2^s < 2^{s+4}$. Thus we have $2^s < 2^r < 2^{s+4}$ which gives $r = s + 1$ or $s + 2$ or $s + 3$. Since 2^r is obtained from 2^s by permuting its digits, $2^r - 2^s$ is divisible by 9. If $r = s + 1$, we see that $2^r - 2^s = 2^s$ and it is clearly not divisible by 9. Similarly, $2^{s+2} - 2^s = 3 \times 2^s$ and $2^{s+3} - 2^s = 7 \times 2^s$ and none of these is divisible by 9. We conclude that $s < r$ is not possible. Hence $r = s$.

4. Is it possible to write the numbers $17, 18, 19, \ldots, 32$ in a 4×4 grid of unit squares, with one number in each square, such that the product of the numbers in each 2×2 sub-grids $AMRG$, $GRND$, $MBHR$ and $RHCN$ is **divisible** by 16?



   Solution: NO! The product of all the numbers in all the four subsquares is divisible by $16 \times 16 \times 16 \times 16 = 2^{16}$. But we also observe that

$$17 \times 18 \times 19 \times \cdots \times 32 = 2^{16}k$$

   where k is an odd integer. Observe that there is $32 = 2^5$ and it must appear in some subsquare. Hence there will be 2^{11} available for the product of the remaining three subsquares. But they must account for $16 \times 16 \times 16 = 2^{12}$. This shortage shows that it is not possible to write 16 numbers in 16 squares such that the product of four numbers in each subsquares is divisible by 16.

5. Let ABC be an acute-angled triangle and let H be its ortho-centre. For any point P on the circum-circle of triangle ABC, let Q be the point of intersection of the line BH with the line AP. Show that there is a unique point X on the circum-circle of ABC such that for every point $P \neq A, B$, the circum-circle of HQP pass through X.

   Solution: We consider two possibilities: Q lying between A and P; and P lying between A and Q. (See the figures.)

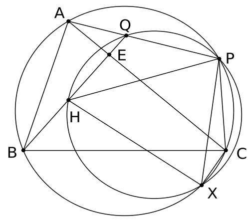
   In the first case, we observe that

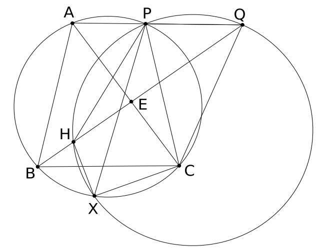
$$\angle HXC = \angle HXP + \angle PXC = \angle AQB + \angle PAC,$$

   since Q, H, X, P are concyclic and P, A, X, C are also concyclic. Thus we get

$$\angle HXC = \angle AQE + \angle QAE = 90^\circ$$

   because $BE \perp AC$.





In the second case, we have

$$\angle HXC = \angle HXP + \angle PXC = \angle HQP + \angle PAC;$$

the first follows from H, X, Q, P are concyclic; the second follows from the concyclicity of A, X, C, P. Again $BE \perp AC$ shows that $\angle HXC = 90^\circ$.

Thus for any point $P \neq A, B$ on the circumcircle of ABC, the point X of intersection of the circumcircles of ABC and HPQ is such that $\angle HXC = 90^\circ$. This means X is precisely the point of intersection of the circumcircles of HEC and ABC, which is independent of P.

6. Let $x_1, x_2, \ldots, x_{2014}$ be positive real numbers such that $\sum_{j=1}^{2014} x_j = 1$. Determine with proof the smallest constant K such that

$$K \sum_{j=1}^{2014} \frac{x_j^2}{1 - x_j} \geq 1.$$

Solution: Let us take the general case: $\{x_1, x_2, \ldots, x_n\}$ are positive real numbers such that $\sum_{k=1}^{n} x_k = 1$. Then

$$\sum_{k=1}^{n} \frac{x_k^2}{1 - x_k} = \sum_{k=1}^{n} \frac{x_k^2 - 1}{1 - x_k} + \sum_{k=1}^{n} \frac{1}{1 - x_k} = \sum_{k=1}^{n} (-1 - x_k) + \sum_{k=1}^{n} \frac{1}{1 - x_k}.$$

Now the first sum is $-n - 1$. We can estimate the second sum using AM-HM inequality:

$$\sum_{k=1}^{n} \frac{1}{1 - x_k} \geq \frac{n^2}{\sum_{k=1}^{n} (1 - x_k)} = \frac{n^2}{n - 1}.$$

Thus we obtain

$$\sum_{k=1}^{n} \frac{x_k^2}{1 - x_k} \geq -(1 + n) + \frac{n^2}{n - 1} = \frac{1}{n - 1}.$$

Here equality holds if and only if all x_j's are equal. Thus we get the smallest constant K such that

$$K \sum_{j=1}^{2014} \frac{x_j^2}{1 - x_j} \geq 1$$

to be $2014 - 1 = 2013$.

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