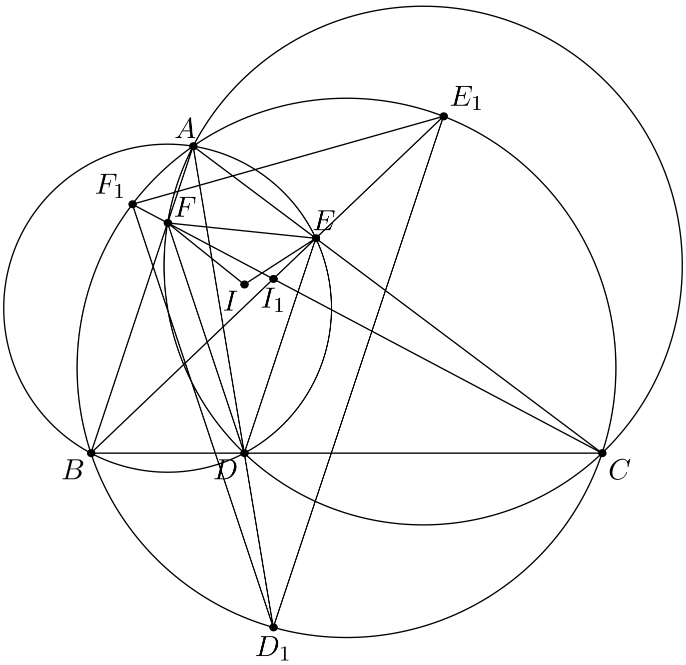
IOQM 2022 Part B

Official Solutions

Problem 1. Let D be an interior point on the side BC of an acute-angled triangle ABC. Let the circumcircle of triangle ADB intersect AC again at $E(\neq A)$ and the circumcircle of triangle ADC intersect AB again at $F(\neq A)$. Let AD, BE and CF intersect the circumcircle of triangle ABC again at $D_1(\neq A)$, $E_1(\neq B)$ and $F_1(\neq C)$, respectively. Let I and I_1 be the incentres of triangles DEF and $D_1E_1F_1$, respectively. Prove that E, F, I, I_1 are concyclic.



Solution. Note that

$$\angle CF_1D_1 = \angle CAD_1 = \angle EAD = \angle EBD = \angle E_1BC = \angle E_1F_1C,$$

so F_1C is the bisector of $\angle D_1E_1F_1$. Similarly, E_1B is the bisector of $\angle D_1E_1F_1$, implying $I_1 = BE_1 \cap CF_1$. Now,

$$\begin{aligned}\angle EDF = \angle EDA + \angle FDA = \angle EBA + \angle FCA \\ = \angle E_1BA + \angle F_1CA = \angle E_1D_1A + \angle F_1D_1A = \angle E_1D_1F_1.\end{aligned}$$

Therefore

$$\angle EIF = 90^\circ + \frac{1}{2}\angle EDF = 90^\circ + \frac{1}{2}\angle E_1D_1F_1 = \angle E_1I_1F_1 = \angle EI_1F,$$

which proves the required concyclicity. $\square$

Problem 2. Find all natural numbers n for which there exists a permutation σ of $1, 2, \ldots, n$ such that

$$\sum_{i=1}^{n} \sigma(i)(-2)^{i-1} = 0.$$

Note: A *permutation* of $1, 2, \ldots, n$ is a bijective function from $\{1, 2, \ldots, n\}$ to itself.

Solution. Suppose that $n \equiv 1 \pmod 3$ and σ a permutation of $1, 2, \ldots, n$. Then

$$\sum_{i=1}^{n} \sigma(i)(-2)^{i-1} \equiv \sum_{i=1}^{n} \sigma(i) = \frac{n(n+1)}{2} \pmod 3,$$

and hence the left-hand side is non-zero.

We now show by induction that if $n \equiv 0$ or $2 \pmod 3$ then there exists a permutation of $1, 2, \ldots, n$ satisfying the given condition.

If $n = 2$ then the permutation given by $\sigma(1) = 2, \sigma(2) = 1$ satisfies the given condition. Similarly, if $n = 3$ then the permutation $\sigma(1) = 2, \sigma(2) = 3, \sigma(3) = 1$ satisfies the given condition.

Suppose that for $n = m$ there exists a permutation σ satisfying the given condition. We consider the permutation τ of $1, 2, \ldots, m+3$ given by $\tau(1) = 2, \tau(2) = 3, \tau(m+3) = 1$ and $\tau(i) = \sigma(i-2) + 3$ for $i = 3, 4, \ldots, m+2$. Then

$$\begin{aligned}\sum_{i=1}^{m+3} \tau(i)(-2)^{i-1} &= 2 - 6 + (-2)^{m+2} + \sum_{i=3}^{m+2} 3 \cdot (-2)^{i-1} \\ &= 2 - 6 + (-2)^{m+2} - 4 \cdot ((-2)^m - 1) = 0\,.\end{aligned}$$

Thus, by induction it follows that for every $n \equiv 0$ or $2 \pmod 3$ there exists a permutation satisfying the given condition. □

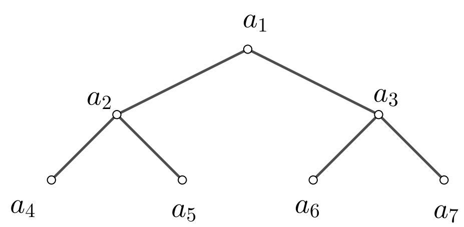
Problem 3. For a positive integer N, let $T(N)$ denote the number of arrangements of the integers $1, 2, \ldots, N$ into a sequence $a_1, a_2, \ldots, a_N$ such that $a_i > a_{2i}$ for all i, $1 \le i < 2i \le N$ and $a_i > a_{2i+1}$, for all i, $1 \le i < 2i + 1 \le N$. For example, $T(3)$ is 2, since the possible arrangements are 321 and 312.

(a) Find $T(7)$.

(b) If K is the largest non-negative integer so that 2^K divides $T(2^n - 1)$, show that $K = 2^n - n - 1$.

(c) Find the largest non-negative integer K so that 2^K divides $T(2^n + 1)$.

Solution. (a) Given an arrangement $a_1, a_2, \ldots, a_7$, satisfying the given conditions, we can build a binary tree with nodes as in the Figure below. At each node, the root node



is greater than the child nodes. Conversely, any such tree gives a valid arrangement. Observing that the root of the tree must contain the maximum of the numbers, we can choose 3 out of the other 6 numbers in $\binom{6}{3}$ ways and build the left tree and the right tree, each in 2 ways and hence the number of such trees is $2 \cdot 2 \cdot \binom{6}{3} = 80$.

(b) Observe that $T(N)$ is also the number of ways of arranging any N distinct numbers into a sequence $a_1, a_2, \ldots, a_N$ satisfying the given conditions. Also, the given conditions imply that $a_1 =$ maximum of the numbers. Now, leaving out the maximum, the rest of the $2^n - 2$ numbers can be split into two groups of $2^{n-1} - 1$ numbers each and these can be individually put into a sequences $b_1, b_2 \ldots, b_{2^{n-1}-1}$ and $c_1, c_2, \ldots, c_{2^{n-1}-1}$ satisfying the conditions in $T(n-1)$ ways each. Now, the required arrangement of the original given sequence can be obtained as follows:

$$a_1, b_1, c_1, b_2, b_3, c_2, c_3, b_4, b_5, b_6, b_7, c_4, c_5, c_6, c_7, \dots$$

This gives

$$T(2^n-1) = T(2^{n-1}-1)^2 \binom{2^n-2}{2^{n-1}-1} \tag{1}$$

We find the highest power of 2 that divides $\binom{2^n-2}{2^{n-1}-1}$:

We have

$$\begin{aligned}
2^{n-2}\binom{2^n}{2^{n-1}} &= 2^{n-2} \cdot \frac{2^n!}{2^{n-1}!2^{n-1}!} \\
&= 2^{n-2} \cdot \frac{2^n(2^n-1)(2^n-2)!}{2^{n-1}(2^{n-1}-1)!2^{n-1}(2^{n-1}-1)!} \\
&= (2^n-1)\binom{2^n-2}{2^{n-1}-1}
\end{aligned}$$

Now, the highest power of 2 that divides $\binom{2^n}{2^{n-1}}$ is

$$\left(2^{n-1}+2^{n-2}+\dots+1\right) - 2\left(2^{n-2}+2^{n-3}+\dots+1\right) = 1$$

Hence the highest power of 2 that divides $\binom{2^n-2}{2^{n-1}-1}$ is $n-1$.

From the recurrence (1), if t_n is the highest power of 2 dividing $T(2^n-1)$, then $t_n = 2t_{n-1}+n-1$. From the initial conditions, $t_1 = 0, t_2 = 1, t_3 = 4$, we obtain, by an easy induction, that $t_n = 2^n - n - 1$.

(c) Suppose that $N = 2^n + 1$. It is easy to see that

$$T(2^n+1) = T(2^{n-1}-1)T(2^{n-1}+1)\binom{2^n}{2^{n-1}+1}$$

The highest power of 2 dividing $\binom{2^n}{2^{n-1}+1}$ is n:

$$(2^{n-1}+1)\binom{2^n}{2^{n-1}+1} = \binom{2^n}{2^{n-1}} \cdot 2^{n-1}$$

Since the highest power of 2 dividing $\binom{2^n}{2^{n-1}}$ is 1, it follows that the highest power of 2 dividing $\binom{2^n}{2^{n-1}+1}$ is n. Thus, if s_n denotes the highest power of 2 dividing $T(2^n+1)$, then

$$s_n = s_{n-1} + 2^{n-1} - (n-1) - 1 + n = s_{n-1} + 2^{n-1}$$

Hence $s_n - s_1 = 2^n - 2$ and since $s_1 = 1$ (since $T(3) = 2$), it follows that the highest power of 2 dividing $T(2^n+1)$ is $2^n - 1$. □