Problem 1. Let $D$ be an interior point on the side $BC$ of an acute-angled triangle $ABC$. Let the circumcircle of triangle $ADB$ intersect $AC$ again at $E(\neq A)$ and the circumcircle of triangle $ADC$ intersect $AB$ again at $F(\neq A)$. Let $AD$, $BE$ and $CF$ intersect the circumcircle of triangle $ABC$ again at $D_1(\neq A)$, $E_1(\neq B)$ and $F_1(\neq C)$, respectively. Let $I$ and $I_1$ be the incentres of triangles $DEF$ and $D_1E_1F_1$, respectively. Prove that $E, F, I, I_1$ are concyclic.

Solution. Note that

$$\angle CF_1D_1 = \angle CAD_1 = \angle EAD = \angle EBD = \angle E_1BC = \angle E_1F_1C,$$

so $F_1C$ is the bisector of $\angle D_1E_1F_1$. Similarly, $E_1B$ is the bisector of $\angle D_1E_1F_1$, implying $I_1 = BE_1 \cap CF_1$. Now,

$$\angle EDF = \angle EDA + \angle FDA = \angle EBA + \angle FCA$$

$$= \angle E_1BA + \angle F_1CA = \angle E_1D_1A + \angle F_1D_1A = \angle E_1D_1F_1.$$

Therefore

$$\angle EIF = 90^\circ + \frac{1}{2} \angle EDF = 90^\circ + \frac{1}{2} \angle E_1D_1F_1 = \angle E_1I_1F_1 = \angle EI_1F,$$

which proves the required concyclicity. \hfill \square

Problem 2. Find all natural numbers $n$ for which there exists a permutation $\sigma$ of $1, 2, \ldots, n$ such that

$$\sum_{i=1}^{n} \sigma(i) (-2)^{i-1} = 0.$$

Note: A permutation of $1, 2, \ldots, n$ is a bijective function from $\{1, 2, \ldots, n\}$ to itself.
Solution. Suppose that \( n \equiv 1 \pmod{3} \) and \( \sigma \) a permutation of \( 1, 2, \ldots, n \). Then
\[
\sum_{i=1}^{n} \sigma(i)(-2)^{i-1} \equiv \sum_{i=1}^{n} \sigma(i) = \frac{n(n+1)}{2} \pmod{3},
\]
and hence the left-hand side is non-zero.

We now show by induction that if \( n \equiv 0 \) or \( 2 \pmod{3} \) then there exists a permutation of \( 1, 2, \ldots, n \) satisfying the given condition.

If \( n = 2 \) then the permutation given by \( \sigma(1) = 2, \sigma(2) = 1 \) satisfies the given condition. Similarly, if \( n = 3 \) then the permutation \( \sigma(1) = 2, \sigma(2) = 3, \sigma(3) = 1 \) satisfies the given condition.

Suppose that for \( n = m \) there exists a permutation \( \sigma \) satisfying the given condition. We consider the permutation \( \tau \) of \( 1, 2, \ldots, m + 3 \) given by \( \tau(1) = 2, \tau(2) = 3, \tau(m + 3) = 1 \) and \( \tau(i) = \sigma(i - 2) + 3 \) for \( i = 3, 4, \ldots, m + 2 \). Then
\[
\sum_{i=1}^{m+3} \tau(i)(-2)^{i-1} = 2 - 6 + (-2)^{m+2} + \sum_{i=3}^{m+2} 3 \cdot (-2)^{i-1} = 2 - 6 + (-2)^{m+2} - 4 \cdot ((-2)^{m} - 1) = 0.
\]

Thus, by induction it follows that for every \( n \equiv 0 \) or \( 2 \pmod{3} \) there exists a permutation satisfying the given condition.

Problem 3. For a positive integer \( N \), let \( T(N) \) denote the number of arrangements of the integers \( 1, 2, \ldots, N \) into a sequence \( a_1, a_2, \ldots, a_N \) such that \( a_i > a_{2i} \) for all \( 1 \leq i < 2i \leq N \) and \( a_i > a_{2i+1} \), for all \( 1 \leq i < 2i + 1 \leq N \). For example, \( T(3) \) is 2, since the possible arrangements are \( 321 \) and \( 312 \).

(a) Find \( T(7) \).

(b) If \( K \) is the largest non-negative integer so that \( 2^K \) divides \( T(2^n - 1) \), show that \( K = 2^n - n - 1 \).

(c) Find the largest non-negative integer \( K \) so that \( 2^K \) divides \( T(2^n + 1) \).

Solution. (a) Given an arrangement \( a_1, a_2, \ldots, a_7 \), satisfying the given conditions, we can build a binary tree with nodes as in the Figure below. At each node, the root node is greater than the child nodes. Conversely, any such tree gives a valid arrangement. Observing that the root of the tree must contain the maximum of the numbers, we can choose 3 out of the other 6 numbers in \( \binom{6}{3} \) ways and build the left tree and the right tree, each in 2 ways and hence the number of such trees is \( 2 \cdot 2 \cdot \binom{6}{3} = 80 \).

(b) Observe that \( T(N) \) is also the number of ways of arranging any \( N \) distinct numbers into a sequence \( a_1, a_2, \ldots, a_N \) satisfying the given conditions. Also, the given conditions imply that \( a_1 = \text{maximum of the numbers} \). Now, leaving out the maximum, the rest of the \( 2^n - 2 \) numbers can be split into two groups of \( 2^{n-1} - 1 \) numbers each and these can be individually put into a sequences \( b_1, b_2, \ldots, b_{2^{n-1} - 1} \) and \( c_1, c_2, \ldots, c_{2^{n-1} - 1} \) satisfying the
conditions in $T(n-1)$ ways each. Now, the required arrangement of the original given sequence can be obtained as follows:

$$a_1, b_1, c_1, b_2, b_3, c_2, b_4, b_5, b_6, c_4, c_5, c_6, c_7, \ldots$$

This gives

$$T(2^n - 1) = T(2^{n-1} - 1)^2 \left( \frac{2^n - 2}{2^{n-1} - 1} \right)$$

(1)

We find the highest power of 2 that divides $\frac{2^n - 2}{2^{n-1} - 1}$:

We have

$$2^{n-2} \left( \frac{2^n}{2^{n-1}} \right) = 2^{n-2} \cdot \frac{2^n}{2^{n-1} \cdot 2^{n-1}}$$

$$= 2^{n-2} \cdot \frac{2^n (2^n - 1)(2^n - 2)!}{2^{n-1}(2^{n-1} - 1)!2^{n-1}(2^{n-1} - 1)!}$$

$$= (2^n - 1) \left( \frac{2^n - 2}{2^{n-1} - 1} \right)$$

Now, the highest power of 2 that divides $\frac{2^n}{2^{n-1}}$ is

$$(2^{n-1} + 2^{n-2} + \ldots + 1) - 2 \left( 2^{n-2} + 2^{n-3} + \ldots + 1 \right) = 1$$

Hence the highest power of 2 that divides $\frac{2^n}{2^{n-1} - 1}$ is $n - 1$.

From the recurrence (1), if $t_n$ is the highest power of 2 dividing $T(2^n - 1)$, then $t_n = 2t_{n-1} + n - 1$. From the initial conditions, $t_1 = 0, t_2 = 1, t_3 = 4$, we obtain, by an easy induction, that $t_n = 2^n - n - 1$.

(c) Suppose that $N = 2^n + 1$. It is easy to see that

$$T(2^n + 1) = T(2^{n-1} - 1)T(2^{n-1} + 1) \left( \frac{2^n}{2^{n-1} + 1} \right)$$

The highest power of 2 dividing $\left( \frac{2^n}{2^{n-1} + 1} \right)$ is $n$:

$$(2^{n-1} + 1) \left( \frac{2^n}{2^{n-1} + 1} \right) = \left( \frac{2^n}{2^{n-1}} \right) \cdot 2^{n-1}$$

Since the highest power of 2 dividing $\left( \frac{2^n}{2^{n-1}} \right)$ is 1, it follows that the highest power of 2 dividing $\left( \frac{2^n}{2^{n-1} + 1} \right)$ is $n$. Thus, if $s_n$ denotes the highest power of 2 dividing $T(2^n + 1)$, then

$$s_n = s_{n-1} + 2^n - (n - 1) - 1 + n = s_{n-1} + 2^n - 1$$

Hence $s_n - s_1 = 2^n - 2$ and since $s_1 = 1$ (since $T(3) = 2$), it follows that the highest power of 2 dividing $T(2^n + 1)$ is $2^n - 1$. 

\[\Box\]