Indian Olympiad Qualifier in Astronomy (IOQA) – 2021-2022

conducted jointly by

Homi Bhabha Centre for Science Education (HBCSE-TIFR)

and

Indian Association of Physics Teachers (IAPT)

Part II: Indian National Astronomy Olympiad (INAO)

Homi Bhabha Centre for Science Education (HBCSE-TIFR)

Question Paper

Date: 20 March 2022 Time: **10:30 – 12:30 hrs**

Roll Number: ☐☐☐☐ - ☐☐☐☐ - ☐☐☐☐☐☐ Maximum Marks: 90

Please Note:

- Please write your roll number in the space provided above.
- There are total 5 questions. Maximum marks are indicated in front of each sub-question.
- For all questions, the process involved in arriving at the solution is more important than the final answer. Valid assumptions / approximations are perfectly acceptable. Please write your method clearly, explicitly stating all reasoning / assumptions / approximations.
- Use of non-programmable scientific calculators is allowed.
- **The answer-sheet must be returned to the invigilator**. You can take this question paper back with you.

Useful Constants

Mass of the Sun	$M_\odot$	$\approx$	1.989×10^{30} kg
Mass of the Earth	$M_\oplus$	$\approx$	5.972×10^{24} kg
Mass of the Moon	$M_☾$	$\approx$	7.347×10^{22} kg
Radius of the Earth	$R_\oplus$	$\approx$	6.371×10^{6} m
Speed of Light	c	$\approx$	$2.998 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Radius of the Sun	$R_\odot$	$\approx$	6.955×10^{8} m
Radius of the Moon	$R_☾$	$\approx$	1.737×10^{6} m
Distance of the Moon from Earth	$d_☾$	$\approx$	3.844×10^{8} m
Astronomical Unit	1 A. U.	$\approx$	1.496×10^{11} m
Univeral Gravitational Constant	G	$\approx$	$6.674 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$
Stefan-Boltzmaan Constant	σ	$\approx$	$5.670 \times 10^{-8}\,\mathrm{Wm^{-2}K^{-4}}$

1. **Transported to an unknown planet**
Imagine that you are kidnapped by some aliens and they drop you on an arbitrary planet in an arbitrary arm of the Milky Way, which does not even have a star to support it. Before leaving you there, the aliens tell you: "Far away from here, in the direction of the star you call Spica, there is a portal which will take you back to the Earth."

You see that the stars appear to go around you once in 25.1 earth hours, around a well-defined axis, which is exactly horizonal at your location. The planet has a tiny moon, which orbits in the planet's equatorial plane with roughly the same angular size at all times. The direction of the moon's revolution is the same as the rotation of the planet. You measure the angular speed of the moon with respect to the distant stars when it is rising and find it to be $2.13 \times 10^{-6}\,\mathrm{rad\,s^{-1}}$. You do the same when the moon is directly overhead and find it to be $1.06 \times 10^{-6}\,\mathrm{rad\,s^{-1}}$.

(a) (5 marks) Use this information to find the revolution period (T_{rev}) of the moon with respect to the distant stars. Assume $d_{\mathrm{m}} \gg R_{\mathrm{p}}$ (d_{m} is the distance to the moon from the centre of the planet and R_{p} is the radius of the planet).

Solution:
If $d_{\mathrm{m}} \gg R_{\mathrm{p}}$, then,

$$\omega_{\mathrm{rising}} = 2.13 \times 10^{-6}\,\mathrm{rad\,s^{-1}}$$
$$\omega_{\mathrm{rising}} = \frac{v}{r} \approx \frac{\omega_{\mathrm{m}} d_{\mathrm{m}}}{d_{\mathrm{m}}} = \omega_{\mathrm{m}} = \frac{2\pi}{T_{\mathrm{rev}}}$$
$$\therefore T_{\mathrm{rev}} = \frac{2\pi}{\omega_{\mathrm{m}}} = \frac{2\pi}{2.13 \times 10^{-6}}$$
$$\boxed{T_{\mathrm{rev}} = 2.95 \times 10^{6}\,\mathrm{s} = 34.14\ \mathrm{days}}$$

(b) (6 marks) Calculate the value of $d_{\mathrm{m}}/R_{\mathrm{p}}$.

Solution:
$$\omega_{\mathrm{p}} = \frac{2\pi}{T_{\mathrm{rot}}} = 6.95 \times 10^{-5}\,\mathrm{rad\,s^{-1}}$$
The angular velocity of moon w.r.t you is
$$\omega_{\mathrm{rising}} = \omega_{\mathrm{m}} = 2.13 \times 10^{-6}\,\mathrm{rad\,s^{-1}}$$
$$\omega_{\mathrm{overhead}} = 1.06 \times 10^{-6}\,\mathrm{rad\,s^{-1}}$$
$$= \frac{\omega_{\mathrm{m}} d_{\mathrm{m}} - \omega_{\mathrm{p}} R_{\mathrm{p}}}{d_{\mathrm{m}}}$$
$$\frac{R_{\mathrm{p}}}{d_{\mathrm{m}}} = \frac{\omega_{\mathrm{m}} - \omega_{\mathrm{overhead}}}{\omega_{\mathrm{p}}}$$
$$\therefore \frac{R_{\mathrm{p}}}{d_{\mathrm{m}}} = \frac{2.13 \times 10^{-6} - 1.06 \times 10^{-6}}{6.95 \times 10^{-5}} = 0.0154$$
$$\boxed{\therefore \frac{d_{\mathrm{m}}}{R_{\mathrm{p}}} = 65.0}$$

(c) (4 marks) You drop some objects on the planet, and realise the acceleration of gravity at the surface to be $7.92\,\mathrm{m\,s^{-2}}$. Estimate the radius (R_p) of the planet.

Solution:

$$\begin{aligned}
GM &= gR_\mathrm{p}^2 \\
\omega_\mathrm{m} &= \sqrt{\frac{GM}{d_\mathrm{m}^3}} \\
\therefore GM &= \omega_\mathrm{m}^2 d_\mathrm{m}^3 \\
\therefore \frac{d_\mathrm{m}^3}{R_\mathrm{p}^2} &= \frac{g}{\omega_\mathrm{m}^2} \\
&= R_\mathrm{p}\left(\frac{d_\mathrm{m}}{R_\mathrm{p}}\right)^3 \\
\therefore R_\mathrm{p} &= \frac{g}{\omega_\mathrm{m}^2} \times \left(\frac{R_\mathrm{p}}{d_\mathrm{m}}\right)^3 = \frac{7.92 \times (0.0154)^3}{(2.13 \times 10^{-6})^2} \\
&\boxed{R_\mathrm{p} = 6.38 \times 10^6\,\mathrm{m}}
\end{aligned}$$

(d) (2 marks) Find the mass (M_p) of the planet.

Solution:

$$\begin{aligned}
M_\mathrm{p} &= \frac{gR_\mathrm{p}^2}{G} \\
&\boxed{M_\mathrm{p} = 4.83 \times 10^{24}\,\mathrm{kg}}
\end{aligned}$$

(e) (3 marks) To escape from this planet, you build a spacecraft, which can carry 1% of its own weight. What is the minimum energy (E_min) needed to reach the portal using this spacecraft?

Solution:
Since you want to escape from the planet, the energy needed will be same as the gravitational potential energy at the planet surface.
Taking some reasonable mass for yourself (40 kg to 80 kg),

$$\begin{aligned}
E_\mathrm{min} &= \frac{GMm}{R_\mathrm{p}} = \frac{6.674 \times 10^{-11} \times 4.83 \times 10^{24} \times 101 \times 60}{6.38 \times 10^6} \\
&\boxed{E_\mathrm{min} = 3.06 \times 10^{11}\,\mathrm{J}}
\end{aligned}$$

Note: This gives an escape velocity of $10.1\,\mathrm{km\,s^{-1}}$. The rotational velocity at equator is $\omega_\mathrm{p} R_\mathrm{p} = 0.5\,\mathrm{km\,s^{-1}}$. It maybe neglected. Solutions including this correction will also be acceptable.

You launch yourself up, when you see Spica correctly positioned. Once you are through the portal you will be back at the exam hall and you may continue solving further.

2. **Radiation pressure**
Consider two stationary identical uniform solid spheres of mass m and radius r placed in a vacuum. Their centres are distance d apart ($d \gg r$). Assume that the spheres are ideal black bodies. Each body is maintained at a temperature T and the system is placed in a large perfectly insulated box. The walls of the box are maintained at 0 K.

   (a) (6 marks) Find the force F that the spheres exert on each other.

Solution:
The two forces at play here are gravity and radiation pressure:

$$\begin{aligned}
F_g &= \frac{-Gm^2}{d^2} \\
F_{RP} &= \frac{E_{\text{incident}}}{ct} = \frac{L_{\text{sphere}}}{c} \times \frac{\pi r^2}{4\pi d^2} \\
&= \left(\frac{4\pi r^2 \sigma T^4}{c}\right) \times \left(\frac{\pi r^2}{4\pi d^2}\right) \\
&= \left(\frac{\pi\sigma}{c}\right) \times \left(\frac{rT}{\sqrt{d}}\right)^4 \\
\therefore F &= F_{RP} + F_g \\
&\boxed{F = \frac{\pi\sigma(rT)^4}{d^2 c} - \frac{Gm^2}{d^2}}
\end{aligned}$$

   (b) (2 marks) Find the temperature T_c at which the net force is exactly zero.

Solution:
Solving for the temperature where $F = 0$ we get:

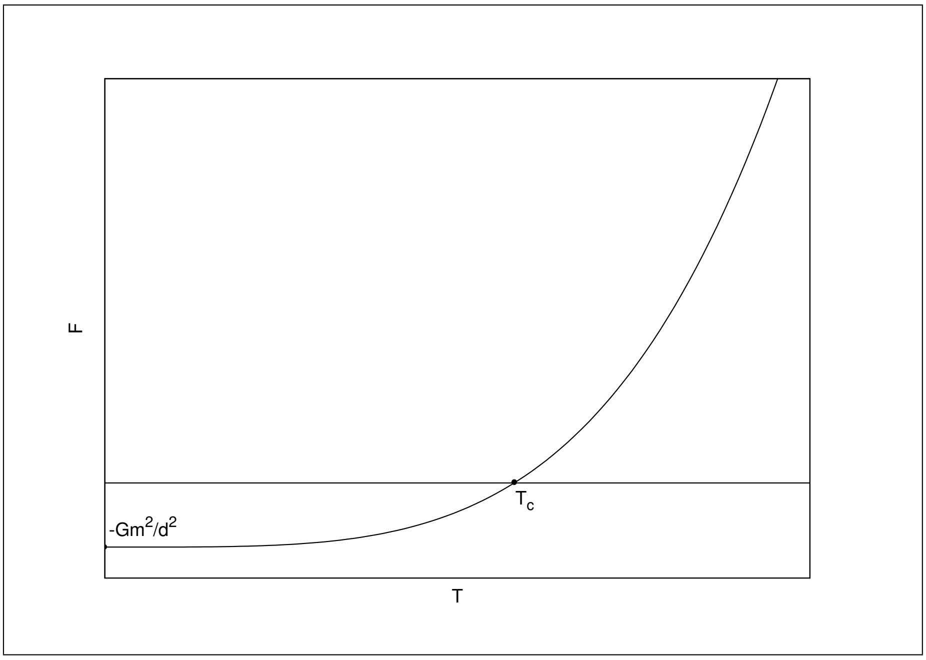
$$\frac{Gm^2}{d^2} = \frac{\pi\sigma(rT_c)^4}{d^2 c}$$

$$\boxed{\therefore T_c = \sqrt[4]{\frac{Gm^2 c}{\sigma\pi r^4}}}$$

   (c) (12 marks) Sketch the graphs F vs T, T_c vs d, $\log(T_c)$ vs $\log(m)$. In each, indicate intercepts on axes and slopes wherever appropriate.

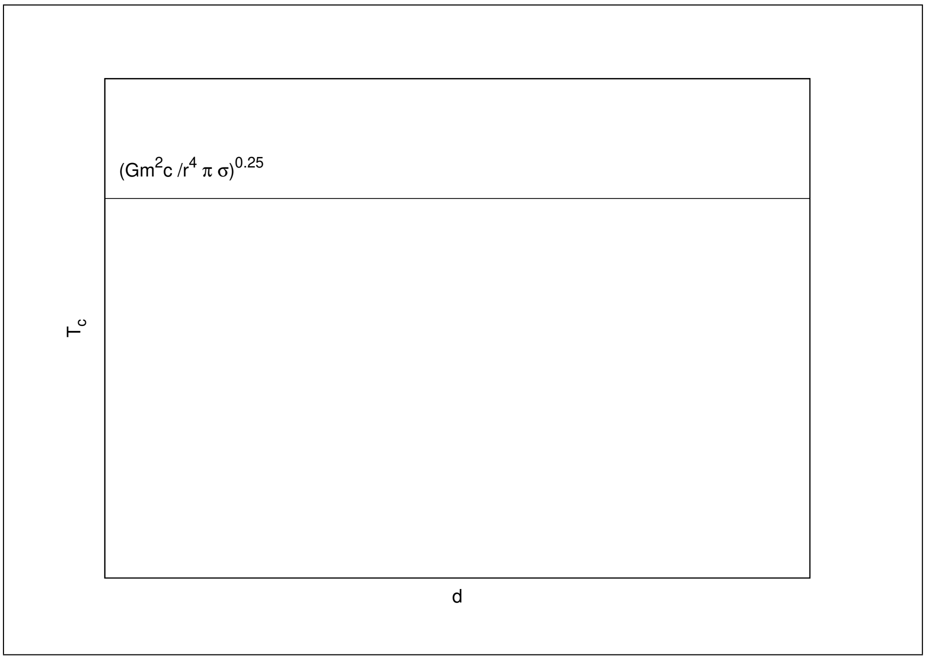
Solution:
<u>F **vs** T</u>:
F vs T will be the same shape as a graph of x^4 centred on the y axis, starting from below the origin by an amount that should be mentioned. Its zero will be T_c which

must be mentioned as well.



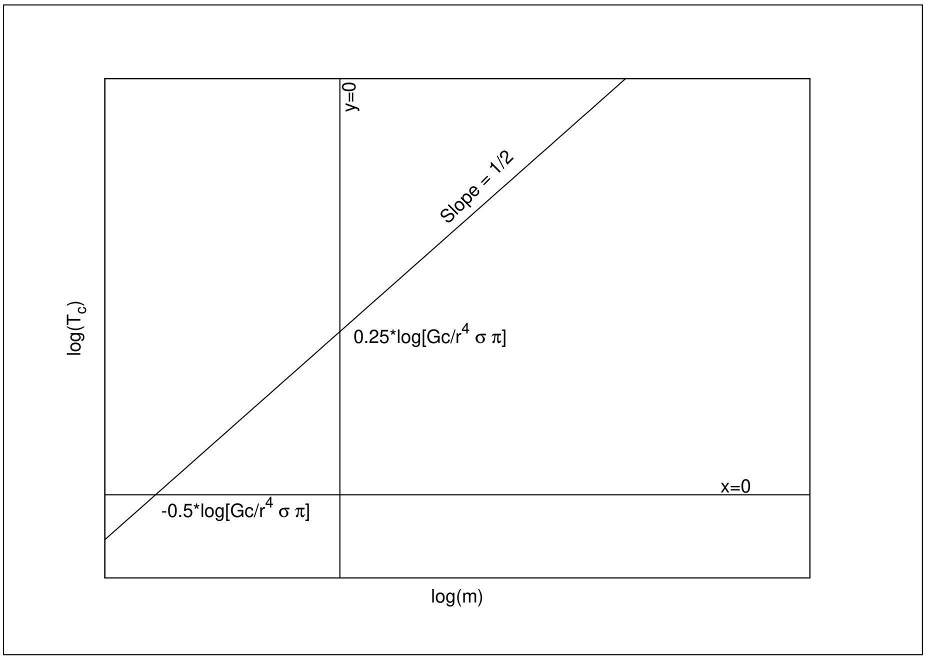
T_c **vs** d**:**

Note that the critical temperature is independent of the separation between the bodies; thus, the T_c vs d graph will be a constant graph, the value of which should be mentioned.



log(T_c**) vs** $\log(m)$**:**

The log-log plot for T_c vs m should have a slope of 0.5 and the intercept should be appropriately mentioned.



Note, all non-log-log plots must have only positive values on axes representing distance, mass or temperature.

(d) (3 marks) Estimate the force (F_{net}) between two 0.5 kg balls of radius 5.0 cm and at $T = 300\,\text{K}$, placed 10 m away from each other. Is the net force attractive or repulsive?

Solution:

$$F_{\text{net}} = \frac{1}{d^2}\left(\frac{\pi\sigma(rT)^4}{c} - Gm^2\right)$$
$$= \frac{1}{10^2}\left(\frac{\pi \times 5.670 \times 10^{-8} \times (0.05 \times 300)^4}{2.998 \times 10^8} - 6.674 \times 10^{-11} \times 0.5^2\right)$$
$$\boxed{F_{\text{net}} \approx 1.3 \times 10^{-13}\,\text{N}}$$

Since the net force is positive, it is a repulsive force.

(e) (2 marks) In his historic experiment to measure the value of the universal gravitational constant G, Lord Cavendish used a similar setup with almost same values of the parameters. However, there was one crucial difference, which enabled him to measure G without worrying about the radiation pressure. What could be that difference?

Solution:

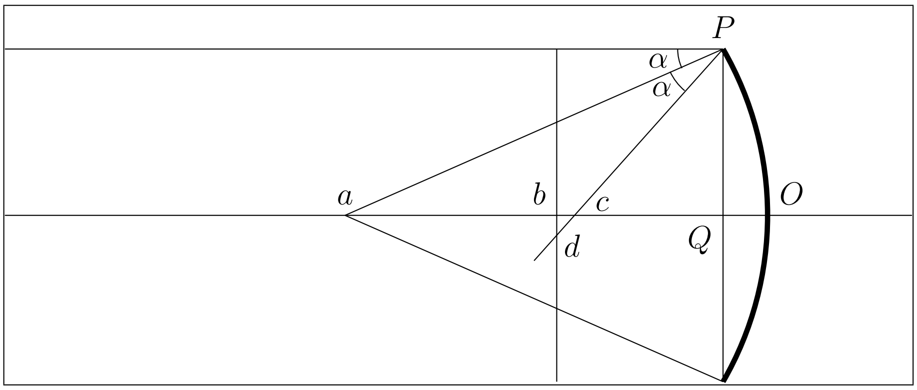
Lord Cavendish used balls of similar size and mass and was able to accurately estimate G. This is because the system that Cavendish had was not in a box maintained at 0 K, but in an environment at equilibrium with the balls. Thus, the blackbody radiation from this environment also fell on the balls from all directions except the direction of the other ball (which shadowed the background radiation). Thus, adding the radiation from the other ball, the radiation force is net zero and only gravity remains in the equation.

3. **Spherical aberration**
A spherical concave mirror gives a real image of a distant object with some spherical aberration. Consider a concave spherical mirror with radius of curvature R. The radius of its rim is r.

(a) (12 marks) For monochromatic light rays coming from infinity and parallel to the optic axis, calculate the size of the image (I_s) at the focal plane (taken to be at $R/2$ distance away from the centre of the mirror) if $R = 1500\,\mathrm{mm}$ and $r = 75\,\mathrm{mm}$.

Solution:
From figure,

$$PQ = r$$
$$aO = R$$
$$bO = \frac{R}{2} = f$$
$$\measuredangle PaQ = \alpha$$
$$\measuredangle PcQ = \measuredangle bcd = 2\alpha$$
$$\sin\alpha = \frac{r}{R}$$



and the image size is $2bd$, where

$$bd = bc\tan 2\alpha$$
$$bc = \frac{R}{2} - cQ - OQ$$
$$OQ = R - \sqrt{R^2 - r^2}$$
$$cQ = \frac{r}{\tan 2\alpha}$$
$$\tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha} = \frac{2\sin\alpha\cos\alpha}{1 - 2\sin^2\alpha} = \frac{2\sin\alpha\sqrt{1-\sin^2\alpha}}{1 - 2\sin^2\alpha}$$
$$\tan 2\alpha = \frac{\frac{2r}{R}\sqrt{1 - \frac{r^2}{R^2}}}{1 - 2\frac{r^2}{R^2}}$$

$$\tan 2\alpha = \frac{2r\sqrt{R^2 - r^2}}{R^2 - 2r^2}$$

bc can be simplified as

$$\begin{aligned}
bc &= \frac{R}{2} - \frac{r}{\tan 2\alpha} - R + \sqrt{R^2 - r^2} \\
&= -\frac{R}{2} - \frac{R^2 - 2r^2}{2\sqrt{R^2 - r^2}} + \sqrt{R^2 - r^2} \\
&= \frac{1}{2}\left(-R - \frac{(R^2 - 2r^2 - 2R^2 + 2r^2)}{\sqrt{R^2 - r^2}}\right) \\
&= \frac{1}{2}\left(\frac{R^2 - R\sqrt{R^2 - r^2}}{\sqrt{R^2 - r^2}}\right)
\end{aligned}$$

Therefore, the image size (I_s) is

$$\begin{aligned}
I_\mathrm{s} &= 2bd \\
&= 2bc\tan 2\alpha \\
&= 2 \times \frac{1}{2}\left(\frac{R^2 - R\sqrt{R^2 - r^2}}{\sqrt{R^2 - r^2}}\right) \times \frac{2r\sqrt{R^2 - r^2}}{R^2 - 2r^2} \\
&= \frac{2rR(R - \sqrt{R^2 - r^2})}{R^2 - 2r^2} \\
&= \frac{2 \times 75 \times 1500 \times (1500 - \sqrt{1500^2 - 75^2})}{1500^2 - 2 \times 75^2}
\end{aligned}$$

$$\boxed{I_\mathrm{s} = 0.189\,\mathrm{mm}}$$

(b) (3 marks) What limit does this spherical abberation place on the angular resolution (θ_min)?

Solution:
Using the relation,

$$l = f\theta$$

where l is the image size, f is the focus of the mirror and θ is the angular size.

$$\begin{aligned}
l &= 2bd = \frac{2rR(R - \sqrt{R^2 - r^2})}{R^2 - 2r^2} \\
f &= \frac{R}{2} \\
\therefore \theta_\mathrm{min} &= \frac{l}{f} = \frac{4r(R - \sqrt{R^2 - r^2})}{R^2 - 2r^2} \\
&= 2.51 \times 10^{-4}\,\mathrm{rad}
\end{aligned}$$

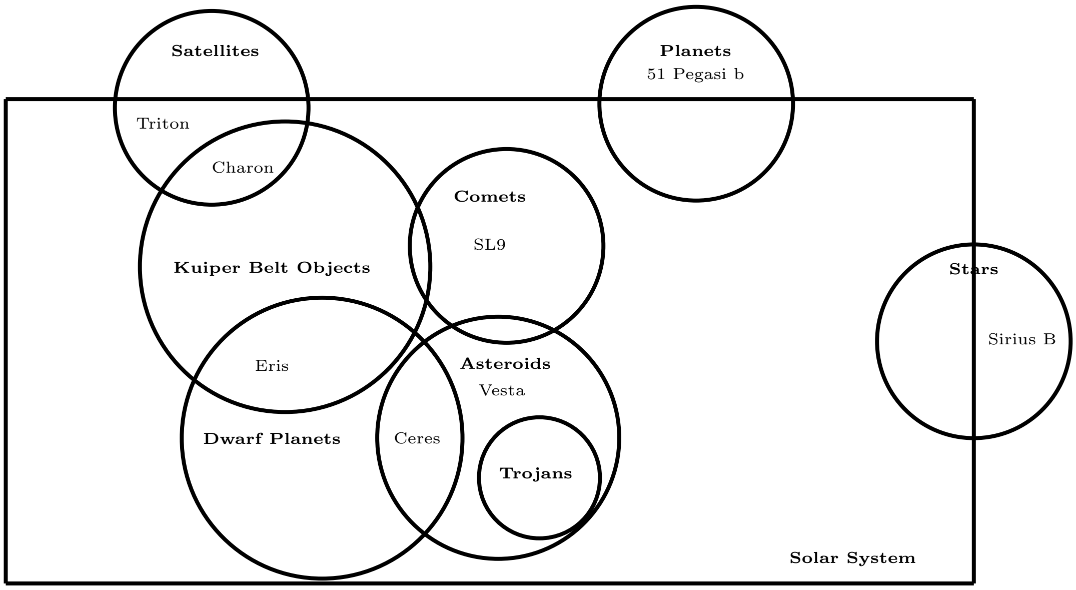
$$\boxed{\theta_\mathrm{min} \approx 51.9''}$$

4. **Venn Diagram**
Here is a list of categories of astronomical objects:
Stars, Planets, Dwarf planets, Asteroids, Kuiper belt objects, Trojans, Comets, Satellites, Solar System Objects

   (a) (8 marks) Draw a Venn diagram that correctly represents these sets.

   (b) (8 marks) Read the list of objects given below. In the Venn diagram that you have drawn, put each of these objects in appropriate section.
   Sirius B, Vesta, Ceres, Charon, Eris, Triton, Shoemaker Levy 9 (SL9), 51 Pegasi b

Solution:



5. The Earth's orbit intersects the orbits of several comets around the Sun. As the Earth passes through such an intersection point on a particular day of the year ("peak date"), the trail of debris left behind by the comet produces a large number of meteors over a few hours. These events are known as "meteor showers". The tracks of all the meteors appear to diverge from a fixed point in the sky, called the "radiant point". The shower is named according to the constellation in which this radiant point lies.

   (a) (7 marks) The radiant points of 7 prominent meteor showers are indicated by the triangles and labeled as letters A to G in the skymap below. Identify these showers from among the ones listed in the table and write the label in the corresponding row (you may ignore the showers that are not shown on the map). You need not write any justification.

These meteor showers occur when the Earth crosses the trail of debris left behind by some comet in the ecliptic plane. Clearly, this is correlated to the Sun's position at that time. The vertical lines in the given skymap indicate the Right Ascension (RA) – a celestial coordinate (similar to longitudes in geography), expressed in "hours" ranging from 0h to 24h, that is used to specify the position of an object in the sky. Distant stars have fixed RA, but close objects like the Sun change their RA due to the revolution of the Earth. The Sun's RA is 0h at the vernal (spring) equinox, and increases over the year, sweeping out the entire range, as the Sun passes through the zodiacal constellations. For example, the Sun lies in the Scorpio constellation in November.

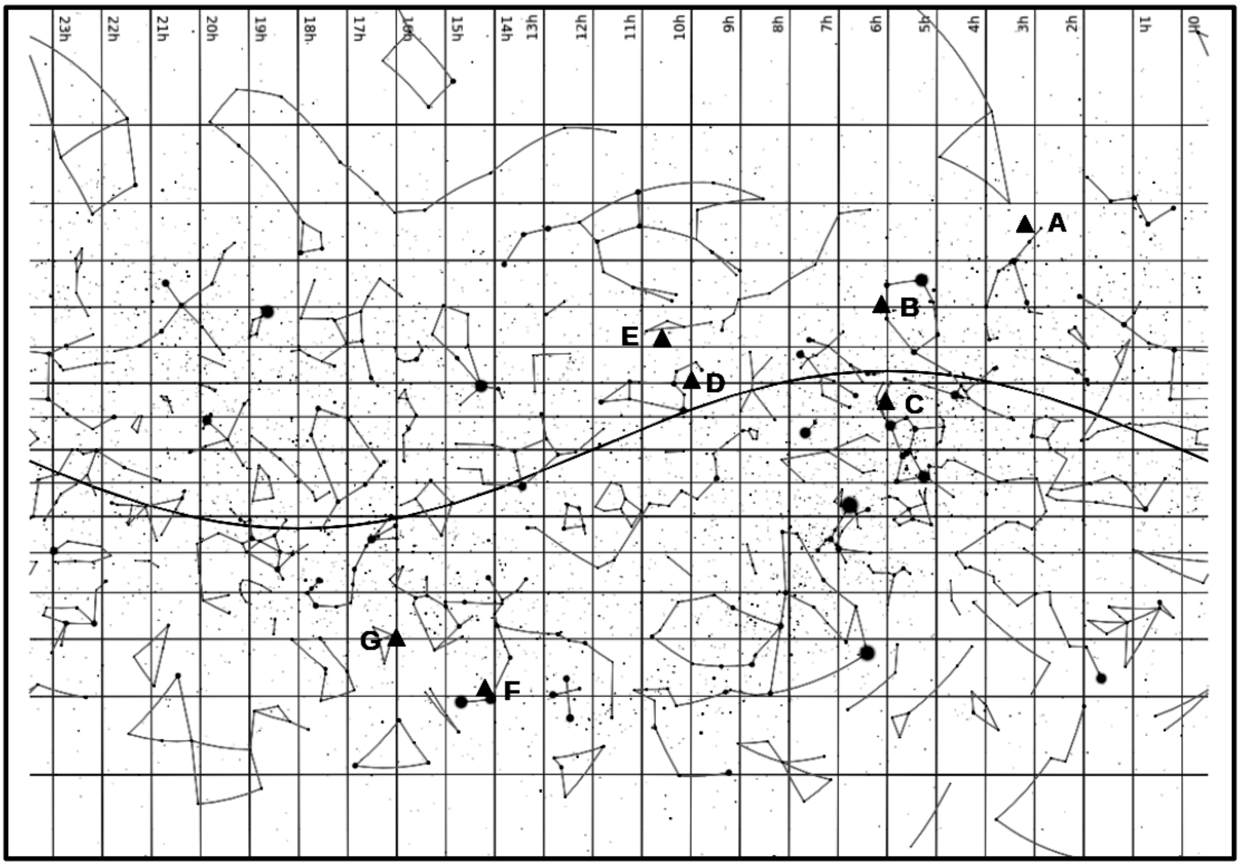


Figure 1: Here filled circles/points represents stars, filled triangles represents radiants of meteor showers and the sinusoidal curve represents the yearly path of Sun (Ecliptic).

(b) (7 marks) Using the above information, identify the peak dates of the 7 meteor showers from among the list given below and write in the given table. You need not write any justification.
List of dates of the peaks of 7 meteor showers: 12 Aug, 1 Sep, 21 Oct, 17 Nov, 19 Dec, 8 Feb, 14 Mar.

Meteor Shower Names	**Alphabetic Label**	**Peak Date**
Alpha Capricornids		
Alpha Centaurids		
Aurigids		
December Leonis Minorids		
Draconids		
Gamma Normids		
Gamma Ursae Minorids		
Geminids		
Leonids		
Lyrids		
Orionids		
Perseids		

Solution:

Meteor Shower Names	Alphabetic Label	Peak Date
Alpha Capricornids		
Alpha Centaurids	F	8 February
Aurigids	B	1 September
December Leonis Minorids	E	19 December
Draconids		
Gamma Normids	G	14 March
Gamma Ursae Minorids		
Geminids		
Leonids	D	17 November
Lyrids		
Orionids	C	21 October
Perseids	A	12 August