35$^{\text{th}}$ Indian National Mathematical Olympiad-2020

Time: 4 hours **January 19, 2020**

Instructions:

- **Calculators (in any form) and protractors are not allowed.**
- **Rulers and compasses are allowed.**
- **All questions carry equal marks. Maximum marks: 102.**
- **Answer all the questions.**
- **Answer to each question should start on a new page. Clearly indicate the question number.**

1. Let Γ_1 and Γ_2 be two circles of unequal radii, with centres O_1 and O_2 respectively, in the plane intersecting in two distinct points A and B. Assume that the centre of each of the circles Γ_1 and Γ_2 is outside the other. The tangent to Γ_1 at B intersects Γ_2 again in C, different from B; the tangent to Γ_2 at B intersects Γ_1 again in D, different from B. The bisectors of $\angle DAB$ and $\angle CAB$ meet Γ_1 and Γ_2 again in X and Y, respectively, different from A. Let P and Q be the circumcentres of triangles ACD and XAY, respectively. Prove that PQ is the perpendicular bisector of the line segment O_1O_2.

2. Suppose $P(x)$ is a polynomial with real coefficients satisfying the condition

$$P(\cos\theta + \sin\theta) = P(\cos\theta - \sin\theta),$$

for every real θ. Prove that $P(x)$ can be expressed in the form

$$P(x) = a_0 + a_1(1-x^2)^2 + a_2(1-x^2)^4 + \cdots + a_n(1-x^2)^{2n},$$

for some real numbers $a_0, a_1, a_2, \ldots, a_n$ and nonnegative integer n.

3. Let $X = \{0,1,2,3,4,5,6,7,8,9\}$. Let $S \subseteq X$ be such that any positive integer n can be written as $p+q$ where the non-negative integers p, q have all their digits in S. Find the smallest possible number of elements in S.

4. Let $n \geq 3$ be an integer and let $1 < a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n$ be n real numbers such that $a_1 + a_2 + a_3 + \cdots + a_n = 2n$. Prove that

$$a_1a_2\cdots a_{n-1} + a_1a_2\cdots a_{n-2} + \cdots + a_1a_2 + a_1 + 2 \leq a_1a_2\cdots a_n.$$

5. Infinitely many equidistant parallel lines are drawn in the plane. A positive integer $n \geq 3$ is called *frameable* if it is possible to draw a regular polygon with n sides all whose vertices lie on these lines and no line contains more than one vertex of the polygon.

   (a) Show that $3, 4, 6$ are *frameable.*

   (b) Show that any integer $n \geq 7$ is not frameable.

   (c) Determine whether 5 is *frameable.*

6. A *stromino* is a 3×1 rectangle. Show that a 5×5 board divided into twenty-five 1×1 squares cannot be covered by 16 *strominos* such that each *stromino* covers exactly three unit squares of the board and every unit square is covered by either one or two *strominos.* (A *stromino* can be placed either horizontally or vertically on the board.)

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