35\textsuperscript{th} Indian National Mathematical Olympiad-2020

Time: 4 hours

January 19, 2020

Instructions:

- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- All questions carry equal marks. Maximum marks: 102.
- Answer all the questions.
- Answer to each question should start on a new page. Clearly indicate the question number.

1. Let $\Gamma_1$ and $\Gamma_2$ be two circles of unequal radii, with centres $O_1$ and $O_2$ respectively, in the plane intersecting in two distinct points $A$ and $B$. Assume that the centre of each of the circles $\Gamma_1$ and $\Gamma_2$ is outside the other. The tangent to $\Gamma_1$ at $B$ intersects $\Gamma_2$ again in $C$, different from $B$; the tangent to $\Gamma_2$ at $B$ intersects $\Gamma_1$ again in $D$, different from $B$. The bisectors of $\angle DAB$ and $\angle CAB$ meet $\Gamma_1$ and $\Gamma_2$ again in $X$ and $Y$, respectively, different from $A$. Let $P$ and $Q$ be the circumcentres of triangles $ACD$ and $XAY$, respectively. Prove that $PQ$ is the perpendicular bisector of the line segment $O_1O_2$.

2. Suppose $P(x)$ is a polynomial with real coefficients satisfying the condition

$$P(cos \theta + sin \theta) = P(cos \theta - sin \theta),$$

for every real $\theta$. Prove that $P(x)$ can be expressed in the form

$$P(x) = a_0 + a_1(1-x^2)^2 + a_2(1-x^2)^4 + \cdots + a_n(1-x^2)^{2n},$$

for some real numbers $a_0, a_1, a_2, \ldots, a_n$ and nonnegative integer $n$.

3. Let $X = \{0,1,2,3,4,5,6,7,8,9\}$. Let $S \subseteq X$ be such that any positive integer $n$ can be written as $p+q$ where the non-negative integers $p,q$ have all their digits in $S$. Find the smallest possible number of elements in $S$.

4. Let $n \geq 3$ be an integer and let $1 < a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n$ be $n$ real numbers such that $a_1 + a_2 + a_3 + \cdots + a_n = 2n$. Prove that

$$a_1a_2\cdots a_{n-1} + a_1a_2\cdots a_{n-2} + \cdots + a_1a_2 + a_1 + 2 \leq a_1a_2\cdots a_n.$$

5. Infinitely many equidistant parallel lines are drawn in the plane. A positive integer $n \geq 3$ is called \textit{frameable} if it is possible to draw a regular polygon with $n$ sides all whose vertices lie on these lines and no line contains more than one vertex of the polygon.

(a) Show that 3, 4, 6 are \textit{frameable}.

(b) Show that any integer $n \geq 7$ is not \textit{frameable}.

(c) Determine whether 5 is \textit{frameable}.

6. A \textit{stromino} is a $3 \times 1$ rectangle. Show that a $5 \times 5$ board divided into twenty-five $1 \times 1$ squares cannot be covered by 16 \textit{strominos} such that each \textit{stromino} covers exactly three unit squares of the board and every unit square is covered by either one or two \textit{strominos}. (A \textit{stromino} can be placed either horizontally or vertically on the board.)