Indian National Astronomy Olympiad – 2020

Question Paper

INAO – 2020

Roll Number: ☐☐☐☐ - ☐☐☐☐ - ☐☐☐☐

Date: 1st February 2020

Duration: **Three Hours**

Maximum Marks: 100

Please Note:

- Please write your roll number in the space provided above.
- There are a total of 7 questions. Maximum marks are indicated in front of each sub-question.
- For all questions, the process involved in arriving at the solution is more important than the final answer. Valid assumptions / approximations are perfectly acceptable. Please write your method clearly, explicitly stating all reasoning / assumptions / approximations.
- Use of non-programmable scientific calculators is allowed.
- **The answer-sheet must be returned to the invigilator**. You can take this question paper back with you.
- Please take note of following details about Orientation-Cum-Selection Camp (OCSC) in Astronomy:
  - Tentative Dates: 21st April to 8th May 2020.
  - This camp will be held at HBCSE, Mumbai.
  - Attending the camp for the entire duration is mandatory for all participants.

Useful Constants

Mass of the Sun	$\mathrm{M}_\odot$	$\approx$	1.989×10^{30} kg
Mass of the Earth	$M_\oplus$	$\approx$	$5.972 \times 10^{24}\,\mathrm{kg}$
Mass of the Moon	M_m	$\approx$	$7.347 \times 10^{22}\,\mathrm{kg}$
Radius of the Earth	$R_\oplus$	$\approx$	$6.371 \times 10^{6}\,\mathrm{m}$
Speed of Light	c	$\approx$	$2.998 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Radius of the Sun	$R_\odot$	$\approx$	$6.955 \times 10^{8}\,\mathrm{m}$
Radius of the Moon	R_m	$\approx$	$1.737 \times 10^{6}\,\mathrm{m}$
Astronomical Unit (au)	$a_\oplus$	$\approx$	$1.496 \times 10^{11}\,\mathrm{m}$
Solar Luminosity	$L_\odot$	$\approx$	$3.826 \times 10^{26}\,\mathrm{W}$
Gravitational Constant	G	$\approx$	$6.674 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$
Gravitational acceleration	g	$\approx$	$9.8\ \mathrm{m/s^2}$
1 parsec (pc)	$1\,\mathrm{pc}$	$=$	$3.086 \times 10^{16}\,\mathrm{m}$
Stefan's Constant	σ	$=$	$5.670 \times 10^{-8}\,\mathrm{Wm^{-2}K^{-4}}$

1. (10 marks) On the evening of an autumnal equinox day Siddhant noticed that Mars was exactly along the north-south meridian in his sky at the exact moment when the sun was setting. In other words, the Sun and Mars subtended an angle of exactly 90° as measured from the Earth. If the orbital radius of Mars is 1.52 au, What will be the approximate rise time of the mars on the next autumnal equinox day?

Solution:

For Earth, $T_\oplus = 1\,\text{yr}$ and $r_\oplus = 1\,\text{au}$
By Kepler's third Law,

$$\begin{aligned} T^2 &\propto r^3 \\ \therefore \left(\frac{T_{Mars}}{T_\oplus}\right)^2 &= \left(\frac{r_{Mars}}{r_\oplus}\right)^3 \\ \therefore T_{Mars} &= (1.52)^{1.5} \\ &= 1.874\,\text{yr} \end{aligned}$$

Thus, in one year, with respect to the Sun, Mars will travel through angle of

$$\frac{1}{1.874} \times 360° = 192.1°$$

Initial M-S-E angle was,

$$\cos^{-1}\left(\frac{1}{1.52}\right) = 48.86°$$

As Mars is to the east of the Sun at the start, it will move towards the opposition and after one year it will be on the west of the Sun.
Thus, the new M-S-E angle (to the west of Sun) will be,

$$192.1° - 48.9° = 143.2°$$

We will apply cosine rule to find earth-mars distance and then apply sine rule to find the angle sutended at the earth.

$$\begin{aligned} d &= \sqrt{1.0^2 + 1.52^2 - 2 \times 1.0 \times 1.52 \cos(143.2°)} \\ &= 2.397\,\text{au} \\ \therefore \theta \sin^{-1} &\left(\frac{\sin(143.2°)}{2.397} \times 1.52\right) \\ &= 22.30° \end{aligned}$$

Now, as the earth completes one rotation in approximately 24 h, The angle of 22.3° corresponds to

$$\frac{22.3°}{360°} \times 24 \approx 89\,\text{min}$$

On equinox day, the day length is exactly 12 h. This means the sun rises at 06:00 local time. Thus, Mars will rise 89 min before that. i.e. it will rise at approximately 04:30 local time.

2. Manoj was determined to claim some world record. He got an idea from the fairy tale of Rapunzel that he will never cut his hair and he can claim world record for growing longest hair.

   (a) (3 marks) Estimate maximum length of the hair that he can grow in his whole life, if he hasn't cut his hair from his birth.

   (b) (7 marks) What will be mass of these hair? (density of typical hair strand is $1.3\,\mathrm{g/cm^3}$)

If you make any simplifying assumptions, discuss qualitatively how answer would have been affected if those assumptions were not made.

Solution:

(a) Considering average human hair grows 1 cm to 2.5 cm every month (one can estimate this by typical length of hair strands cut during regular visits to barber), and average life time of human is about 80 years.
$\therefore$ In the life time of average human total length of hair that a person can grow is (taking average growth of 1.5 cm per month)

$$\boxed{\mathrm{L} = 1.5 \times 12 \times 80 = 1440\,\mathrm{cm} \approx 14\,\mathrm{m}}$$

This assumes the rate of hair growth remains constant and there is no breakage. In reality, longer hairs grow slower and after a certain length they are more prone to breakage.

(b) On trying to measure thickness of hair using a scale, we see that the hair strand is much thinner than the least count of the scale (0.1 cm). Thus rough estimate of the thickness of hair strand would be 0.01 cm.
If density of typical hair strand is $1.3\,\mathrm{g/cm^3}$, then a single strand of human hair weighs about $10^{-2}\,\mathrm{g/m}$.
The space between hair strands must be more than thickness of strands (0.01 cm), but less than least count of scale (0.1 cm).
so rough estimate for space between strands would be 0.05 cm.
Thus atleast one hair strand will be there in a square of 0.05 cm side length, which means one hair strand per $0.0025\,\mathrm{cm^2}$ means roughly 400 hair strands in $1\,\mathrm{cm^2}$
Assuming average human head as a sphere of radius 8 cm, and about one third of it is covered in hair.
$\therefore$ roughly total number of hair strands on average human head would be

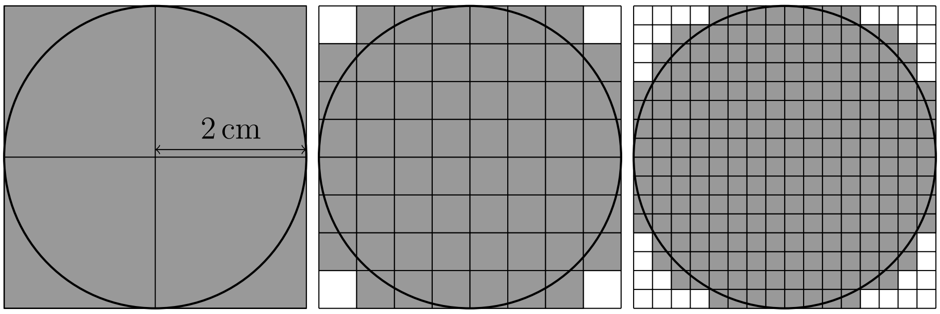
$$N = 4\pi(8)^2 \times \frac{1}{3} \times 400 = 107233 \approx 10^5$$

$\therefore$ total weight of these long hairs would be

$$\boxed{W = 10^5 \times 14 \times 10^{-2} = 14 \times 10^3\,\mathrm{g} \approx 14\,\mathrm{kg}}$$

In most cases, estimation will lead to 1 kg of hair for every metre of length.

3. (10 marks) A curious middle school student wants to actually measure perimeter of a circle of radius 2 cm. He has a scale and plenty of time at his disposal, but does not have a thread to measure lengths along arcs. Hence, he comes up with the following strategy:



1. He makes a circumscribing square of the circle.
2. He divides the square into 4 equal smaller squares (left panel).
3. He measures the outer boundary (perimeter) of a shape formed by all the squares that overlap (fully or partially) with the circle (shape is highlighted in gray in the figure). He calls this estimate as P_1.
4. He then makes a finer grid by dividing each smaller square into 4x4 grid. He again repeats step 3 to estimate the perimeter and calls it P_2 (middle panel).
5. He keeps refining the grid again and again n times and finds final estimate P_n.

Estimate P_n.

Hint: Eliminate white squares one by one, starting from corners, to arrive at the perimeter of gray shape.

Solution:

Take upper left white square in the middle panel. Let us say the side of this square is x cm. Thus, this square contributes length $2x$ to the outer perimeter. Remaining $2x$ length is along inner gray squares.
Now if this piece is taken away, outer perimeter's $2x$ contribution is reduced, but the now exposed gray squares had $2x$ length along the removed piece, which gets added to the outer perimeter. Thus, the outer perimeter remains unchanged.
We can use the same argument to remove other white squares. Hence, the outer perimeter of gray shape, at any stage, is the same as the perimeter of the circumscribing square.

$$\therefore P_1 = P_2 = P_3 = ... = P_n = 16\,\text{cm}$$

4. Let us consider three stars A, B and C. It was observed that

- As seen from star B, star A is barely visible to the naked eye,
- As seen from star C, star B is barely visible to the naked eye,
- As seen from star A, star C is barely visible to the naked eye,

Let us denote the distance between star A and star B as d_1, and distance between star B and C as d_2 and star C and A as d_3.
Absolute magnitude of star A, i.e. $M_A = 2.00\,\mathrm{mag}$ and that of star B is $M_B = 3.00\,\mathrm{mag}$.
(For explanation of magnitude system, see the box below.)

(a) (4 marks) Find the distances d_1 and d_2.

(b) (5 marks) Find the interval (in magnitudes) in which the absolute magnitude M_C has to belong so that the above described configuration is allowed.

(c) (3 marks) If $M_C = 4.00\,\mathrm{mag}$, find the largest angle γ in this stellar triangle.

(d) (6 marks) Show that if we change the values of the three absolute magnitudes so that their differences remain the same, the angles in the triangle will not change.

A note about magnitude system:

The brightness of a star as seen by some observer, is dependent on the observer's distance from the star as well the intrinsic brightness of the star (ignoring any absorption in the intervening space).
In astronomy, brightness of a star, as seen by some observer, is measured in terms of its 'apparent magnitude' (m). For two stars (1 and 2) with fluxes f_1 and f_2 respectively, their apparent magnitudes m_1 and m_2 are related by

$$m_1 - m_2 = -2.5 \times log_{10}\left(\frac{f_1}{f_2}\right)$$

Absolute magnitude (M) of any star is its apparent magnitude, if the star was exactly 10 pc away from the observer. Thus, M only depends on the intrinsic brightness of the star.
The relation between m and M is given as,

$$m - M = -5 + 5\ log_{10}(d)$$

where d is measured in parsec (pc).
By convention, the faintest stars visible to naked eye, in ideal viewing conditions, have been assigned an apparent magnitude of $m = +6.0\,\mathrm{mag}$.

Solution:

(a) Star A is barely visible from B, which implies that the apparent magnitude of star A as seen from star B is 6 mag, (i.e. $m_A = 6\,\mathrm{mag}$) , hence using above relation,

$$\begin{aligned} m_A - M_A &= -5 + 5\ log_{10}(d_1) \\ \therefore d_1 &= 10^{\left(\frac{6-2+5}{5}\right)} = 10^{1.8} = 63.09\,\mathrm{pc} \end{aligned}$$

Similarly,

$$\begin{aligned} m_B - M_B &= -5 + 5\ log_{10}(d_2) \\ \therefore d_2 &= 39.81\,\mathrm{pc} \end{aligned}$$

(b) Using triangle inequality, if a, b and c are the three sides of the triangle then,

$$
\begin{aligned}
&|a-b| \leq c \leq a+b \\
&\therefore d_3 \leq d_1 + d_2 = 102.9\,\mathrm{pc} \\
&\Rightarrow M_C \geq 0.93\,\mathrm{mag} \\
&\text{or } d_3 \geq d_1 - d_2 = 23.29\,\mathrm{pc} \\
&\Rightarrow M_C \leq 4.17\,\mathrm{mag}
\end{aligned}
$$

Range for M_C is 0.93 to 4.17 mag.

(c) Absolute magnitude of star C is given to be $M_C = 4\,\mathrm{mag}$. Thus,

$$d_3 = 25.12\,\mathrm{pc}$$

Since d_1 is largest, angle opposite will be the largest.
Using the cosine rule, we can write,

$$
\begin{aligned}
&d_1^2 = d_2^2 + d_3^2 - 2d_2d_3\cos C \\
&\Rightarrow C = 152^\circ
\end{aligned}
$$

(d) Let us subtract the equations in pairs, we will get,

$$
\begin{aligned}
m_A - M_A &= -5 + 5\ log_{10}(d_1) \\
m_B - M_B &= -5 + 5\ log_{10}(d_2) \\
\text{As } m_A = m_B &= 6.0\,\mathrm{mag} \\
-M_A + M_B &= 5\ log_{10}\left(\frac{d_1}{d_2}\right) \\
\text{Similarly,} & \\
-M_B + M_C &= 5\ log_{10}\left(\frac{d_2}{d_3}\right) \\
-M_{A'} + M_{B'} &= 5\ log_{10}\left(\frac{d_{1'}}{d_{2'}}\right) \\
-M_{B'} + M_{C'} &= 5\ log_{10}\left(\frac{d_{2'}}{d_{3'}}\right)
\end{aligned}
$$

Now, the condition states that, differences in the three absolute magnitudes remains constant, therefore we may write,

$$\begin{aligned}
M_A \; - \; M_B &= \; M_{A'} \; - \; M_{B'} \\
M_B \; - \; M_C &= \; M_{B'} \; - \; M_{C'} \\
\Rightarrow \; 5 \; log_{10} \left(\frac{d_1}{d_2}\right) &= \; 5 \; log_{10} \left(\frac{d_{1'}}{d_{2'}}\right) \\
\left(\frac{d_1}{d_2}\right) &= \; \left(\frac{d_{1'}}{d_{2'}}\right)
\end{aligned}$$

Similarly,

$$\left(\frac{d_2}{d_3}\right) = \left(\frac{d_{2'}}{d_{3'}}\right)$$

We also know from sine rule that,

$$\begin{aligned}
\frac{\sin A}{d_2} &= \frac{\sin B}{d_3} = \frac{\sin C}{d_1} \\
\frac{d_1}{d_2} &= \frac{\sin C}{\sin A} \\
\frac{d_2}{d_3} &= \frac{\sin A}{\sin B}
\end{aligned}$$

From the above relations, it is clear that changing the absolute magnitudes, in a way that their differences remain the same, will not change the angles of this stellar triangle.

5. An extremely powerful cannon is placed horizontally at the north pole (at ground level) and its barrel is aligned with the Prime Meridian (Greenwich Meridian). The cannon is fired at $t = 0$, and the cannonball has sufficient velocity to travel in a circular orbit around the Earth. Assume the Earth to be a perfect sphere and that no objects obstruct the path of the cannonball.

   (a) (3 marks) What is the period of the orbit of this cannonball?

   (b) (10 marks) Plot the trajectory of the cannonball (latitude (ϕ) vs. longitude (λ)) on a regular graph paper given in your answersheet. The plot should correspond to one full orbit.

   (c) (7 marks) The second graph given in the answersheet is a part of a large polar projection plot having concentric arcs around the centre as latitudes and radial lines as longitudes. Define appropriate scale and show the path of cannonball on this graph from the moment it is fired till the moment it crosses the equator.

Solution:

(a)

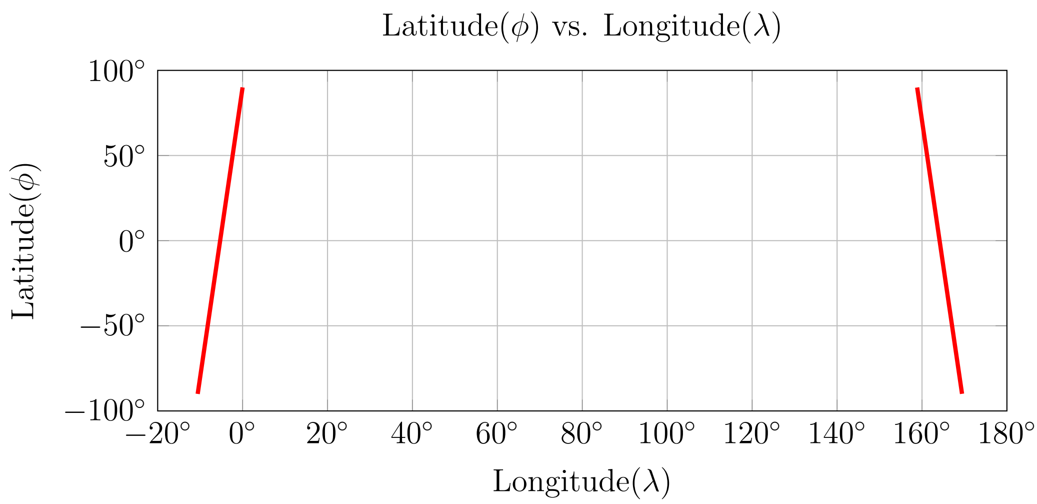
$$\text{Circular Orbital velocity} = V = \sqrt{\frac{GM}{R}}$$
$$= 7909.86\,\text{m/s}$$
$$\text{Orbital time period} = T = \frac{2\pi R}{V}$$
$$= 5060.63\,\text{s} = 84.34\,\text{min}$$

(b) The change in latitude (λ) is due to the Circular angular velocity (ω_λ) and the change in longitude (ϕ) is due to the Earth's angular velocity (ω_ϕ).

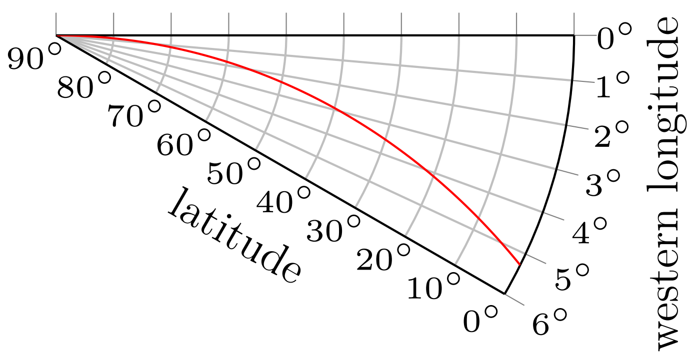
$$\omega_\lambda = \frac{360}{84.34} = 4.27\,^\circ/\text{min}$$
$$\omega_\phi = \frac{360}{24 \times 60} = 0.25\,^\circ/\text{min}$$

Since both the angular velocities are constant the graph will be linear and the slope will be:

$$\text{slope} = \frac{\omega_\lambda}{\omega_\phi} = 17.08$$



(c) The nature of the graph is given below.
The tangent of curve at the pole must have slope=0 and the direction of the tangent must be in the direction of the meridian.



6. Pulsars are a type of fast rotating, high density stars, which are known for their regular pulses of radiation. They are also sometimes called as the 'light houses of the universe'. In this problem, we assume the pulsars to be spheres of uniform density, which are gravitationally bound.

   (a) (4 marks) Rotation period of a pulsar was measured to be $P = 1.500\,\mathrm{ms}$. What limit does this put on its density?

   (b) (2 marks) If mass of this pulsar is $1.5\mathrm{M}_\odot$, what limit can we place on its radius?

   A pulsar is formed from a massive progenitor star which typically has a magnetic field of 0.1 T and average density of $10^{-4}\,\mathrm{g\,cm^{-3}}$. Such a star loses about 90% of its mass towards the end of its life. The remnant mass forms a pulsar of the kind described above.

   (c) (2 marks) Along the lines of part (a), what limit does this put on the rotation period of the progenitor star?

   (d) (4 marks) Assuming that the magnetic flux is conserved during the formation of a pulsar, find the typical magnetic field at the surface of the pulsar?

Solution:

(a) For a particle of mass (m) on the surface of the pulsar of mass M_P and radius r, the condition to stay attached to the pulsar will be

$$
\begin{aligned}
mr\omega^2 &\leq \frac{GM_\mathrm{P}m}{r^2} \\
\omega^2 &\leq \frac{GM_\mathrm{P}}{r^3} = \frac{4\pi G\rho_p}{3} \\
\therefore \rho_p &\geq \frac{3\omega^2}{4\pi G} = \frac{3\pi}{GT^2} \\
\rho_p &\geq 6.28 \times 10^{16}\,\mathrm{kg\,m^{-3}}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\rho_p &= \frac{3M_\mathrm{P}}{4\pi\ r^3} \\
r &\leq 22.5\,\mathrm{km}
\end{aligned}
$$

(c) By centrifugal force criterion,

$$
\begin{aligned}
\omega^2 &\leq \frac{4\pi G\rho}{3} \\
\omega &\leq 5.29 \times 10^{-6}\,\mathrm{rad\,s^{-1}} \\
\therefore t &\geq \frac{2\pi}{\omega} = 13.8\,\mathrm{days}
\end{aligned}
$$

Thus, we can say rotation period of the star is slower than 14 days.

(d) We assume that during the collapse of inner layers, magnetic flux is conserved. Thus, the value of magnetic field will increase in the ratio of the surface areas, before and after the collapse.

$$\frac{B_{\rm st}}{B_{\rm P}} = \frac{r_{\rm P}^2}{r_{\rm st}^2}$$

$$\text{Now, } r_{\rm st} = \sqrt[3]{\frac{3\ M}{4\ \pi\ \rho_{\rm st}}} = \sqrt[3]{\frac{30\ M_{\rm P}}{4\ \pi\ \rho_{\rm st}}}$$

$$\therefore r = 4.15 \times 10^{10}\,\mathrm{m} = 59.3\mathrm{R}_\odot$$

$$\therefore B_{\rm P} = B_{\rm st} \times \frac{r_{\rm st}^2}{r_{\rm P}^2}$$

$$B_{\rm P} = 3.40 \times 10^{11}\,\mathrm{T}$$

The magnetic field on the surface of a pulsar is of the order of 10^8 to 10^{10} T.

7. The picture below was taken on 24th December 2019, from some place in India, showing the crescent of the Moon near the horizon (the horizontal dashed line marks the horizon of the place). Field of View (FOV) of the image is 60°.

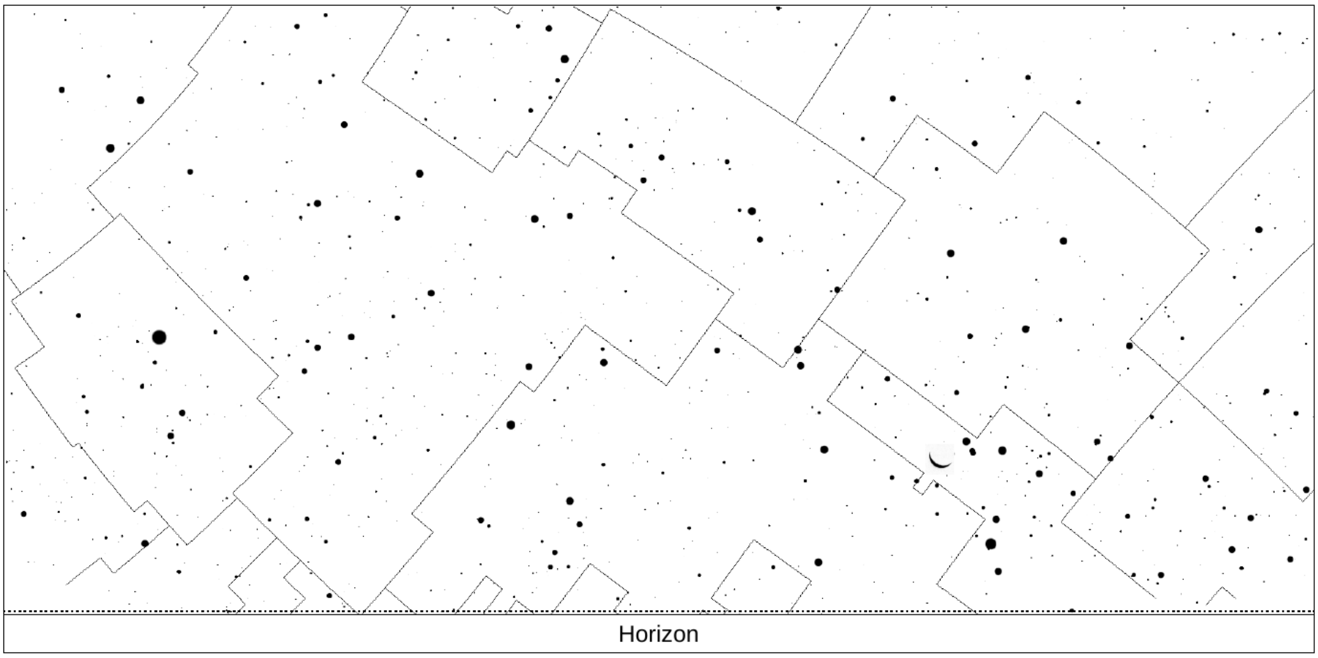


Figure 1: Negative image of certain patch of the sky on 24th December 2019

Note: The images printed are colour-inverted, i.e. the bright parts of the image appear black and dark parts appear white. Thus, the black dots are stars and planet and the dark cresent is actually bright crescent of the moon.

(a) (5 marks) At which of the following times this picture may have been taken? Give justification for your answer.
18:00 hrs, 22:00 hrs, 01:00 hrs, 05:00 hrs

(b) (5 marks) Write the names of the constellations present in the image of the sky.

(c) (1 mark) The map also includes a planet. Mark the planet on the map with a circle and label as 'P'.

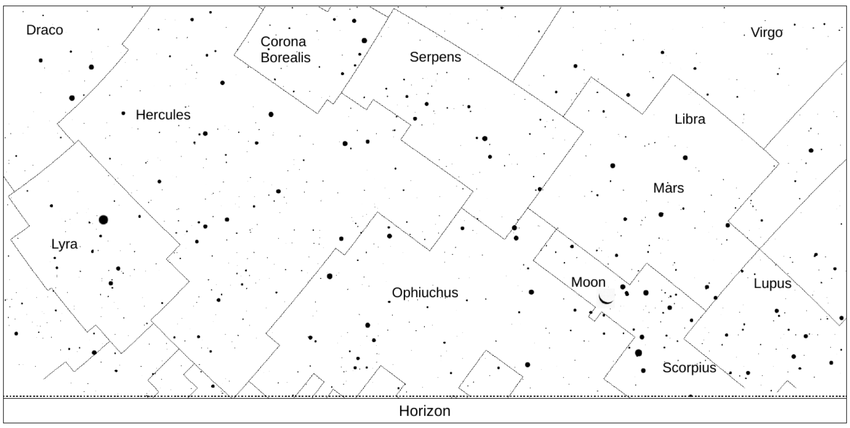
(d) (5 marks) A zoomed-in image of the lunar crescent is given in the answersheet. On this image mark the approximate directions to the cardinal points. [East-West-North-South] Note: You may assume the box in the answersheet has a linear angular scale along the horizon.

(e) (4 marks) Find approximate latitude of the place.

Solution:

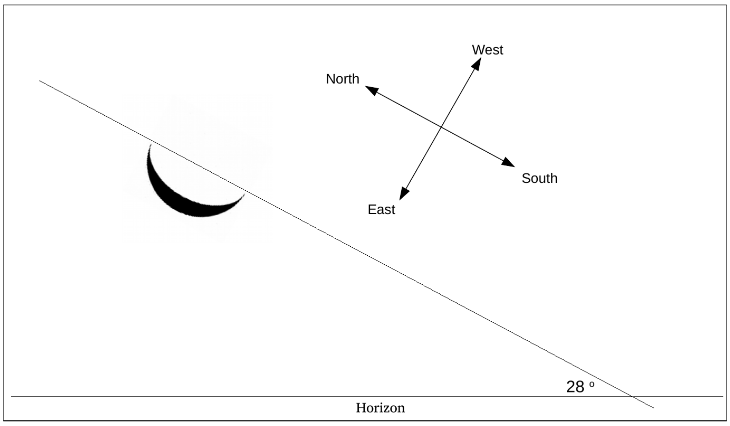
(a) On 26th December 2019, there was an annular solar eclipse, which can happen only on a New Moon day. On the New Moon day Sun and Moon rise together. Every day Moon rises 50 minutes late. Therefore on 24th November, i.e. 2 days before New Moon, the moon would rise at approximately 4 am in the morning. Hence we can say that the picture was taken in the morning.

(b) Following constellations are present in the sky,
Moon is in Scorpius, Libra, Ophiuchus, Lyra, Corona Borealis, Hercules, Serpens also Lupus, Virgo and Draco are partly seen.
The brightest object in Libra is Mars.



(c) The line connecting the two tips of the crescent gives us a line parallel to the North-South line. Thus, the East-West direction would be perpendicular to this line.
The directions should not be marked on the horizon but should be shown as arrows (see figure.)

(d) When the Moon is sufficiently close to the horizon, we can approximate the surface of sky as a simple plane. Hence, the angle made by the tips of the crescent with respect to horizon will approximately give you the latitude of the place.


West
North
South
East
28 °
Horizon