Regional Mathematical Olympiad-2000
Problems and Solutions

1. Let AC be a line segment in the plane and B a point between A and C. Construct isosceles triangles PAB and QBC on one side of the segment AC such that $\angle APB = \angle BQC = 120°$ and an isosceles triangle RAC on the otherside of AC such that $\angle ARC = 120°$. Show that PQR is an equilateral triangle.

   Solution: We give here 2 different solutions.

   1. Drop perpendiculars from P and Q to AC and extend them to meet AR, RC in K, L respectively. Join KB, PB, QB, LB, KL.(Fig.1.)

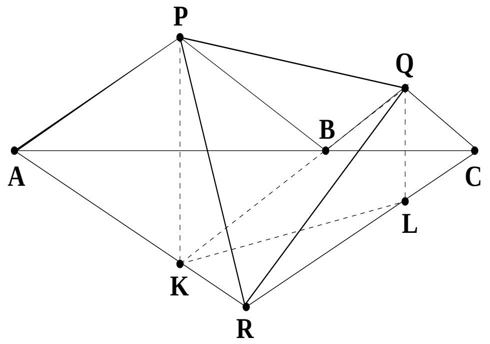


Fig. 1

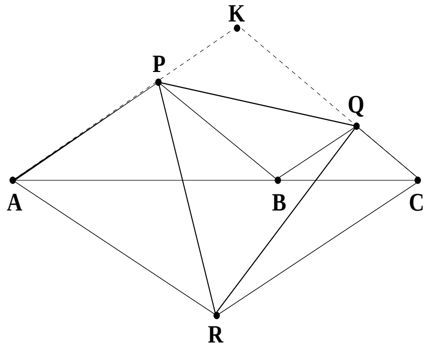


Fig. 2

   Observe that K, B, Q are collinear and so are P, B, L. (This is because $\angle QBC = \angle PBA = \angle KBA$ and similarly $\angle PBA = \angle CBL$.) By symmetry we see that $\angle KPQ = \angle PKL$ and $\angle KPB = \angle PKB$. It follows that $\angle LPQ = \angle LKQ$ and hence K, L, Q, P are concyclic. We also note that $\angle KPL + \angle KRL = 60° + 120° = 180°$. This implies that P, K, R, L are concyclic. We conclude that P, K, R, L, Q are concyclic. This gives

$$\angle PRQ = \angle PKQ = 60°, \quad \angle RPQ = \angle RKQ = \angle RAP = 60°.$$

   2. Produce AP and CQ to meet at K. Observe that $AKCR$ is a rhombus and $BQKP$ is a parallelogram.(See Fig.2.) Put $AP = x, CQ = y$. Then $PK = BQ = y$, $KQ = PB = x$ and $AR = RC = CK = KA = x + y$. Using cosine rule in triangle PKQ, we get $PQ^2 = x^2 + y^2 - 2xy\cos 120° = x^2 + y^2 + xy$. Similarly cosine rule in triangle QCR gives $QR^2 = y^2 + (x+y)^2 - 2xy\cos 60° = x^2 + y^2 + xy$ and cosine rule in triangle PAR gives $RP^2 = x^2 + (x+y)^2 - 2xy\cos 60° = x^2 + y^2 + xy$. It follows that $PQ = QR = RP$.

2. Solve the equation $y^3 = x^3 + 8x^2 - 6x + 8$, for positive integers x and y.

Solution: We have

$$y^3 - (x+1)^3 = x^3 + 8x^2 - 6x + 8 - (x^3 + 3x^2 + 3x + 1) = 5x^2 - 9x + 7.$$

Consider the quadratic equation $5x^2 - 9x + 7 = 0$. The discriminant of this equation is $D = 9^2 - 4 \times 5 \times 7 = -59 < 0$ and hence the expression $5x^2 - 9x + 7$ is positive for all real values of x. We conclude that $(x+1)^3 < y^3$ and hence $x + 1 < y$.

On the other hand we have

$$(x+3)^3 - y^3 = x^3 + 9x^2 + 27x + 27 - (x^3 + 8x^2 - 6x + 8) = x^2 + 33x + 19 > 0$$

for all positive x. We conclude that $y < x + 3$. Thus we must have $y = x + 2$. Putting this value of y, we get

$$0 = y^3 - (x+2)^3 = x^3 + 8x^2 - 6x + 8 - (x^3 + 6x^2 + 12x + 8) = 2x^2 - 18x.$$

We conclude that $x = 0$ and $y = 2$ or $x = 9$ and $y = 11$.

3. Suppose $\langle x_1, x_2, \ldots, x_n, \ldots \rangle$ is a sequence of positive real numbers such that $x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_n \cdots$, and for all n

$$\frac{x_1}{1} + \frac{x_4}{2} + \frac{x_9}{3} + \cdots + \frac{x_{n^2}}{n} \leq 1.$$

Show that for all k the following inequality is satisfied:

$$\frac{x_1}{1} + \frac{x_2}{2} + \frac{x_3}{3} + \cdots + \frac{x_k}{k} \leq 3.$$

Solution: Let k be a natural number and n be the unique integer such that $(n-1)^2 \leq k < n^2$. Then we see that

$$\begin{aligned}
\sum_{r=1}^{k} \frac{x_r}{r} &\leq \left(\frac{x_1}{1} + \frac{x_2}{2} + \frac{x_3}{3}\right) + \left(\frac{x_4}{4} + \frac{x_5}{5} + \cdots + \frac{x_8}{8}\right) \\
&\quad + \cdots + \left(\frac{x_{(n-1)^2}}{(n-1)^2} + \cdots + \frac{x_k}{k} + \cdots + \frac{x_{n^2-1}}{n^2-1}\right) \\
&\leq \left(\frac{x_1}{1} + \frac{x_1}{1} + \frac{x_1}{1}\right) + \left(\frac{x_4}{4} + \frac{x_4}{4} + \cdots + \frac{x_4}{4}\right) \\
&\quad + \cdots + \left(\frac{x_{(n-1)^2}}{(n-1)^2} + \cdots + \frac{x_{(n-1)^2}}{(n-1)^2}\right) \\
&= \frac{3x_1}{1} + \frac{5x_2}{4} + \cdots + \frac{(2n-1)x_{(n-1)^2}}{(n-1)^2} \\
&= \sum_{r=1}^{n-1} \frac{(2r+1)x_{r^2}}{r^2}
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{r=1}^{n-1} \frac{3r}{r^2} x_{r^2} \\
&= 3 \sum_{r=1}^{n-1} \frac{x_{r^2}}{r} \leq 3,
\end{aligned}
$$

where the last inequality follows from the given hypothesis.

4. All the 7-digit numbers containing each of the digits $1, 2, 3, 4,\ 5, 6, 7$ exactly once, and not divisible by 5, are arranged in the increasing order. Find the 2000-th number in this list.

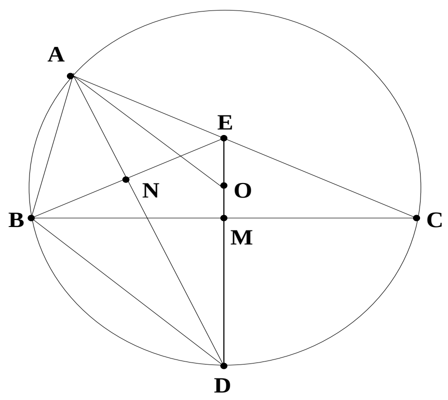
   Solution: The number of 7-digit numbers with 1 in the left most place and containing each of the digits $1, 2, 3, 4,\ 5, 6, 7$ exactly once is $6! = 720$. But 120 of these end in 5 and hence are divisible by 5. Thus the number of 7-digit numbers with 1 in the left most place and containing each of the digits $1, 2, 3, 4,\ 5, 6, 7$ exactly once but not divisible by 5 is 600. Similarly the number of 7-digit numbers with 2 and 3 in the left most place and containing each of the digits $1, 2, 3, 4,\ 5, 6, 7$ exactly once but not divisible by 5 is also 600 each. These account for 1800 numbers. Hence 2000-th number must have 4 in the left most place.

   Again the number of such 7-digit numbers beginning with 41,42 and not divisible by 5 is $120 - 24 = 96$ each and these account for 192 numbers. This shows that 2000-th number in the list must begin with 43.

   The next 8 numbers in the list are: 4312567, 4312576, 4312657, 4312756, 4315267, 4315276, 4315627 and 4315672. Thus 2000-th number in the list is 4315672.

5. The internal bisector of angle A in a triangle ABC with $AC > AB$, meets the circumcircle Γ of the triangle in D. Join D to the centre O of the circle Γ and suppose DO meets AC in E, possibly when extended. Given that BE is perpendicular to AD, show that AO is parallel to BD.

   Solution: We consider here the case when ABC is an acute-angled triangle; the cases when $\angle A$ is obtuse or one of the angles $\angle B$ and $\angle C$ is obtuse may be handled similarly.



Let M be the point of intersection of DE and BC; let AD intersect BE in N. Since ME is the perpendicular bisector of BC, we have $BE = CE$. Since AN is the internal bisector of $\angle A$, and is perpendicular to BE, it must bisect BE; i.e., $BN = NE$. This in turn implies that DN bisects $\angle BDE$. But $\angle BDA = \angle BCA = \angle C$. Thus $\angle ODA = \angle C$. Since $OD = OA$, we get $\angle OAD = \angle C$. It follows that $\angle BDA = \angle C = \angle OAD$. This implies that OA is parallel to BD.

6. (i) Consider two positive integers a and b which are such that $\boldsymbol{a^a b^b}$ is divisible by 2000. What is the least possible value of the product ab?
(ii) Consider two positive integers a and b which are such that $\boldsymbol{a^b b^a}$ is divisible by 2000. What is the least possible value of the product ab?

Solution: We have $2000 = 2^4 5^3$.

(i) Since 2000 divides $a^a b^b$, it follows that 2 divides a or b and similarly 5 divides a or b. In any case 10 divides ab. Thus the least possible value of ab for which $2000 | a^a b^b$ must be a multiple of 10. Since 2000 divides $10^{10} 1^1$, we can take $a = 10, b = 1$ to get the least value of ab equal to 10.

(ii) As in (i) we conclude that 10 divides ab. Thus the least value of ab for which $2000 | a^b b^a$ is again a multiple of 10. If $ab = 10$, then the possibilities are $(a, b) = (1, 10), (2, 5), (5, 2), (10, 1)$. But in all these cases it is easy to verify that 2000 does not divide $a^b b^a$. The next multiple of 10 is 20. In this case we can take $(a, b) = (4, 5)$ and verify that 2000 divides $4^5 5^4$. Thus the least value here is 20.

7. Find all real values of a for which the equation $x^4 - 2ax^2 + x + a^2 - a = 0$ has all its roots real.

Solution: Let us consider $x^4 - 2ax^2 + x + a^2 - a = 0$ as a quadratic equation in a. We see that thee roots are

$$a = x^2 + x, \quad a = x^2 - x + 1.$$

Thus we get a factorisation

$$(a - x^2 - x)(a - x^2 + x - 1) = 0.$$

It follows that $x^2 + x = a$ or $x^2 - x + 1 = a$. Solving these we get

$$x = \frac{-1 \pm \sqrt{1 + 4a}}{2}, \quad \text{or} \quad x = \frac{-1 \pm \sqrt{4a - 3}}{2}.$$

Thus all the four roots are real if and only if $a \geq 3/4$.