34^{th} Indian National Mathematical Olympiad-2019

Problems and Solutions

1. Let ABC be a triangle with $\angle BAC > 90^\circ$. Let D be a point on the segment BC and E be a point on the line AD such that AB is tangent to the circumcircle of triangle ACD at A and BE is perpendicular to AD. Given that $CA = CD$ and $AE = CE$, determine $\angle BCA$ in degrees.

   Solution: Let $\angle C = 2\alpha$. Then $\angle CAD = \angle CDA = 90^\circ - \alpha$. Moreover $\angle BAD = 2\alpha$ as BA is tangent to the circumcircle of $\triangle CAD$. Since $AE = AD$, it gives $\angle AEC = 2\alpha$. Thus $\triangle AEC$ is similar to $\triangle ACD$. Hence
   $$\frac{AE}{AC} = \frac{AC}{AD}.$$
   But the condition that $BE \perp AD$ gives $AE = AB\cos 2\alpha = c\cos 2\alpha$. It is easy to see that $\angle B = 90^\circ - 3\alpha$. Using sine rule in triangle ADC, we get
   $$\frac{AD}{\sin 2\alpha} = \frac{AC}{\sin(90-\alpha)}.$$
   This gives $AD = 2b\sin\alpha$. Thus we get
   $$b^2 = AC^2 = AE \cdot AD = (c\cos 2\alpha)\cdot 2b\sin\alpha.$$
   Using $b = 2R\sin B$ and $c = 2R\sin C$, this leads to
   $$\cos 3\alpha = 2\sin 2\alpha\cos 2\alpha\sin\alpha = \sin 4\alpha\sin\alpha.$$
   Writing $\cos 3\alpha = \cos(4\alpha - \alpha)$ and expanding, we get $\cos 4\alpha\cos\alpha = 0$. Therefore $\alpha = 90^\circ$ or $4\alpha = 90^\circ$. But $\alpha = 90^\circ$ is not possible as $\angle C = 2\alpha$. Therefore $4\alpha = 90^\circ$ which gives $\angle C = 2\alpha = 45^\circ$.

2. Let $A_1B_1C_1D_1E_1$ be a regular pentagon. For $2 \le n \le 11$,
   let $A_nB_nC_nD_nE_n$ be the pentagon whose vertices are the midpoints of the sides of the pentagon $A_{n-1}B_{n-1}C_{n-1}D_{n-1}E_{n-1}$. All the 5 vertices of each of the 11 pentagons are arbitrarily coloured red or blue. Prove that four points among these 55 points have the same colour and form the vertices of a cyclic quadrilateral.

   Solution: We first observe that all the eleven pentagons are regular. Moreover, there are 5 fixed directions and all the 55 sides are in one of these directions. If we consider any two sides which are parallel, they are the parallel sides of an isosceles trapezium, which is cyclic.

   If we consider any pentagon, its two adjacent vertices have the same colour. Consider all such 11 sides whose end points are of the same colour. These are in 5 fixed directions. By pigeon-hole principle, there are 3 sides which are in the same directions and therefore parallel to each other. Among these three sides, two must have end points having one colour (again by P-H principle). Thus there are two parallel sides among the 55 and the end points of these have one fixed colour. But these two sides are parallel sides of an isosceles trapezium. Hence the four end points are concyclic.

3. Let m, n be distinct positive integers. Prove that
   $$\gcd(m,n) + \gcd(m+1,n+1) + \gcd(m+2,n+2) \le 2|m-n| + 1.$$
   Further, determine when equality holds.

Solution: Observe that

$$\gcd(m+j, n+j) = \gcd(m+j, |m-n|),$$

for $j = 0, 1, 2$. Hence we can find positive integers a, b, c such that

$$\gcd(m,n) = \frac{|m-n|}{a}, \quad \gcd(m+1, n+1) = \frac{|m-n|}{b}, \quad \gcd(m+2, n+2) = \frac{|m-n|}{c}.$$

It follows that $|m-n|$ divides ma, $(m+1)b$ and $(m+2)c$. Hence we can see that $|m-n|$ divides ab and bc. We get $|m-n| \le ab$ and $|m-n| \le bc$. This leads to

$$b \ge \frac{|m-n|}{a}, \quad b \ge \frac{|m-n|}{c}.$$

Thus

$$\begin{aligned} &\gcd(m,n) + \gcd(m+1, n+1) + \gcd(m+2, n+2) \\ &\qquad = \frac{|m-n|}{a} + \frac{|m-n|}{b} + \frac{|m-n|}{c} \le 2b + \frac{|m-n|}{b}. \end{aligned}$$

We have to prove that

$$2b + \frac{|m-n|}{b} \le 2|m-n| + 1.$$

Taking $|m-n| = K$, we have to show that $2b^2 + K \le b(2K+1)$. This reduces to $(b-K)(2b-1) \le 0$. However

$$K = |m-n| \ge b \ge 1 > \frac{1}{2}.$$

Equality holds only when $(m,n) = (k, k+1)$ or $(2k, 2k+2)$ or permutations of these for some k.

4. Let n and M be positive integers such that $M > n^{n-1}$. Prove that there are n distinct primes $p_1, p_2, p_3, \ldots, p_n$ such that p_j divides $M + j$ for $1 \le j \le n$.

   Solution: If some number $M+k$, $1 \le k \le n$, has at least n distinct prime factors, then we can associate a prime factor of $M+k$ with the number $M+k$ which is not associated with any of the remaining $n-1$ numbers.

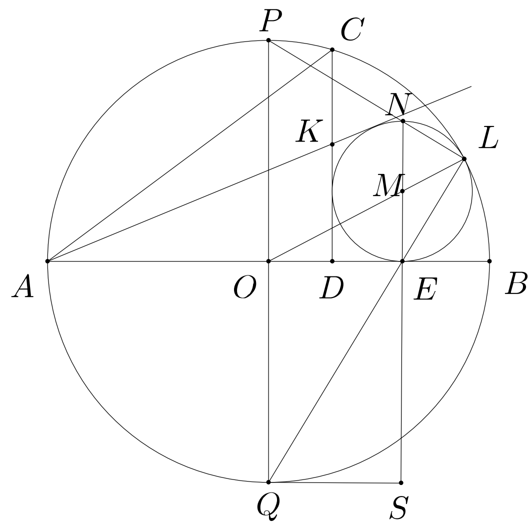
   Suppose $m+j$ has less than n distinct prime factors. Write

   $$M + j = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}, \quad r < n.$$

   But $M + j > n^{n-1}$. Hence there exist t, $1 \le t \le r$ such that $p_t^{\alpha_t} > n$. Associate p_t with this $M+j$. Suppose p_t is associated with some $M+l$. Let $p_t^{\beta_t}$ be the largest power of p_t dividing $M+l$. Then $p_t^{\beta_t} > n$. Let $T = \gcd\left(p_t^{\alpha_t}, p_t^{\beta_t}\right)$. Then $T > n$. Since $T|(M+j)$ and $T|(M+l)$, it follows that $T|(|j-l|)$. But $|j-l| < n$ and $T > n$, and we get a contradiction. This shows that p_t cannot be associated with any other $M+l$. Thus each $M+j$ is associated with different primes.

5. Let AB be a diameter of a circle Γ and let C be a point on Γ different from A and B. Let D be the foot of perpendicular from C on to AB. Let K be a point of the segment CD such that AC is equal to the semiperimeter of the triangle ADK. Show that the excircle of triangle ADK opposite A is tangent to Γ.

   Solution: Draw another diameter $PQ \perp AB$. Let E be the point at which the excircle Γ_1 touches the line AD. Join QE and extend it to meet Γ in L. Draw the diameter EN of Γ_1 and draw $QS \perp NE$ (extended). See the figure. We also observe that $DE = EM = EN/2$.



Since AE is equal to the semiperimeter of $\triangle ADK$, we have $AC = AE$. Hence $AE^2 = AC^2 = AD \cdot AB$ (as ACB is a right-angle triangle). Thus

$$AD(AD + DE + EB) = (AD + DE)^2 = AD^2 + 2AD \cdot DE + DE^2.$$

Simplification gives

$$\begin{aligned} AD \cdot EB &= AD \cdot DE + DE^2 \\ &= DE(AD + DE) \\ &= DE \cdot AE \\ &= DE(AB - BE). \end{aligned}$$

Therefore

$$DE \cdot AB = EB(AD + DE) = EB \cdot AE.$$

But

$$DE \cdot AB = DE \cdot PQ = 2DE \cdot OQ = EN \cdot ES,$$

and $EB \cdot AE = QE \cdot EL$. Therefore we get

$$QE \cdot EL = EN \cdot ES.$$

It follows that Q, S, L, N are concyclic. Since $\angle QSE = 90°$, we get $\angle ELN = 90°$. Since EN is a diameter, this implies that L also lies on Γ_1. But $\angle QLP = 90°$. Therefore L, N, P are collinear. Since $NM \parallel PO$ and

$$\frac{NM}{PO} = \frac{NE}{PQ} = \frac{LN}{LP},$$

it follows that L, M, O are collinear. Hence Γ_1 is tangent to Γ at L.

Alternate solution: Let R be the radius the circle Γ and r be that of the circle Γ_1. Let O be the centre of Γ and M be that of the circle Γ_1. Let E be the point of contact of Γ_1 with AB. Then $ME = DE = r$. Observe that AE is the semiperimeter of $\triangle ADE$. We are given that $AC = AE$. Using that $\angle ACB = 90°$, we also get $AC^2 = AD \cdot AB$. Hence $AE^2 = AD \cdot AB$. We have to show that $R - r = OM$ for proving that Γ_1 is tangent to Γ. We have

$$\begin{aligned} OM^2 - (R-r)^2 = OE^2 + r^2 - (R-r)^2 &= (AD + DE - AO)^2 + r^2 - (R-r)^2 \\ &= (AD - (R-r))^2 + r^2 - (R-r)^2 = AD^2 - 2AD \cdot (R-r) + r^2 \\ &= (AD^2 + 2AD \cdot r + r^2) - 2AD \cdot R = (AD + r)^2 - AD \cdot AB \\ &= (AD + DE)^2 - AD \cdot AB = AE^2 - AD \cdot AB = 0. \end{aligned}$$

Hence $OM = R - r$ and therefore Γ_1 is tangent to Γ.

6. Let f be function defined from the set $\{(x,y) : x, y \text{ reals}, xy \neq 0\}$ in to the set of all positive real numbers such that

$$\begin{aligned} (i) \qquad & f(xy,z) = f(x,z)f(y,z), \text{ for all } x,y \neq 0; \\ (ii) \qquad & f(x,yz) = f(x,y)f(x,z), \text{ for all } x,y \neq 0; \\ (iii) \qquad & f(x,1-x) = 1, \text{ for all } x \neq 0,1. \end{aligned}$$

Prove that

$$\begin{aligned} \text{(a)} \qquad & f(x,x) = f(x,-x) = 1, \text{ for all } x \neq 0; \\ \text{(b)} \qquad & f(x,y)f(y,x) = 1, \text{ for all } x,y \neq 0. \end{aligned}$$

Solution: (The condition (ii) was inadvertently left out in the paper. We give the solution with condition (ii).)

Taking $x = y = 1$ in (ii), weget $f(1,z)^2 = f(1,z)$ so that $f(1,z) = 1$ for all $z \neq 0$. Similarly, $x = y = -1$ gives $f(-1,z) = 1$ for all $z \neq 0$. Using the second condition, we also get $f(z,1) = f(z,-1) = 1$ for all $z \neq 0$. Observe

$$f\left(\frac{1}{x}, y\right) f(x,y) = f(1,y) = 1 = f(x,1) = f\left(x, \frac{1}{y}\right) f(x,y).$$

Therefore

$$f\left(x, \frac{1}{y}\right) = f\left(\frac{1}{x}, y\right) = \frac{1}{f(x,y)},$$

for all $x, y \neq 0$. Now for $x \neq 0, 1$, condition (iii) gives

$$1 = f\left(\frac{1}{x}, 1 - \frac{1}{x}\right) = f\left(x, \frac{1}{1-\frac{1}{x}}\right).$$

Multiplying by $1 = f(x, 1-x)$, we get

$$1 = f(x,1-x) f\left(x, \frac{1}{1-\frac{1}{x}}\right) = f\left(x, \frac{1-x}{1-\frac{1}{x}}\right) = f(x,-x),$$

for all $x \neq 0, 1$. But $f(x,-1) = 1$ for all $x \neq 0$ gives

$$f(x,x) = f(x,-x)f(x,-1) = f(x,-x) = 1$$

for all $x \neq 0, 1$. Observe $f(1,1) = f(1,-1) = 1$. Hence

$$f(x,x) = f(x,-x) = 1$$

for all $x \neq 0$, which proves (a).

We have

$$1 = f(xy,xy) = f(x,xy)f(y,xy) = f(x,x)f(x,y)f(y,x)f(y,y) = f(x,y)f(y,x),$$

for all $x, y \neq 0$, which proves (b).

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