33^{rd} Indian National Mathematical Olympiad-2018

Problems and brief solutions

1. Let ABC be a non-equilateral triangle with integer sides. Let D and E be respectively the mid-points BC and CA; let G be the centroid of triangle ABC. Suppose D, C, E, G are concyclic. Find the least possible perimeter of triangle ABC.

   Solution: Let $m_b = BE$. Then $BG = 2m_b/3$. Since D, C, E, G are concyclic, we know that $BD \cdot BC = BG \cdot BE$. This along with Appolonius' theorem gives

   $$a^2 + b^2 = 2c^2$$

   Since a, b are integers, this implies that a, b must have same parity. This gives

   $$\left(\frac{a-b}{2}\right)^2 + \left(\frac{a+b}{2}\right)^2 = c^2.$$

   Thus $\Big((a-b)/2, (a+b)/2, c\Big)$ is a Pythagorean triplet. Consider the first triplet $(3, 4, 5)$. This gives $a = 7, b = 1$ and $c = 5$. But a, b, c are not the sides of a triangle. The next triple is $(5, 12, 13)$. We obtain $a = 17, b = 7$ and $c = 13$. In this case we get a triangle and its perimeter is $17 + 7 + 13 = 37$. Since $2c < a + b + c < 3c$, it is sufficient to verify up to $c = 19$.

2. For any natural number n, consider a $1 \times n$ rectangular board made up of n unit squares. This is covered by three types of tiles: 1×1 red tile, 1×1 green tile and 1×2 blue domino. Let t_n denote the number of ways of covering $1 \times n$ rectangular board by these three types of tiles. Prove that t_n divides t_{2n+1}.

   Solution: Consider a $1 \times (2n + 1)$ board and imagine the board to be placed horizontally. Let us label the squares of the board as

   $$C_{-n}, C_{-(n-1)}, \ldots, C_{-2}, C_{-1}, C_0, C_1, C_2, \ldots, C_{n-1}, C_n$$

   from left to right. The 1×1 tiles will be referred to as tiles, and the blue 1×2 tile will be referred to as a domino.

   Let us consider the different ways in which the centre square C_0 can be covered. There are four distinct ways in which this can be done:

   (a) There is a blue domino covering the squares C_{-1}, C_0. In this case, there is a $1 \times (n-1)$ board remaining on the left of this domino which can be covered in t_{n-1} ways, and there is a $1 \times n$ board remaining on the right of the domino which can be covered in t_n ways.

   (b) There is a blue domino covering the squares C_0, C_1. In this case, there is a $1 \times n$ board remaining on the left of this domino which can be covered in t_n ways, and there is a $1 \times (n-1)$ board remaining on the right of the domino which can be covered in $t_(n-1)$ ways.

(c) There is a red tile covering the square C_0. In this case, there is a $1 \times n$ board remaining on both sides of this tile, each of which can be covered in t_n ways.

(d) There is a green tile covering the square C_0. In this case, there is a $1 \times n$ board remaining on both sides of this tile, each of which can be covered in t_n ways.

Putting all the possibilities mentioned above together, we get that

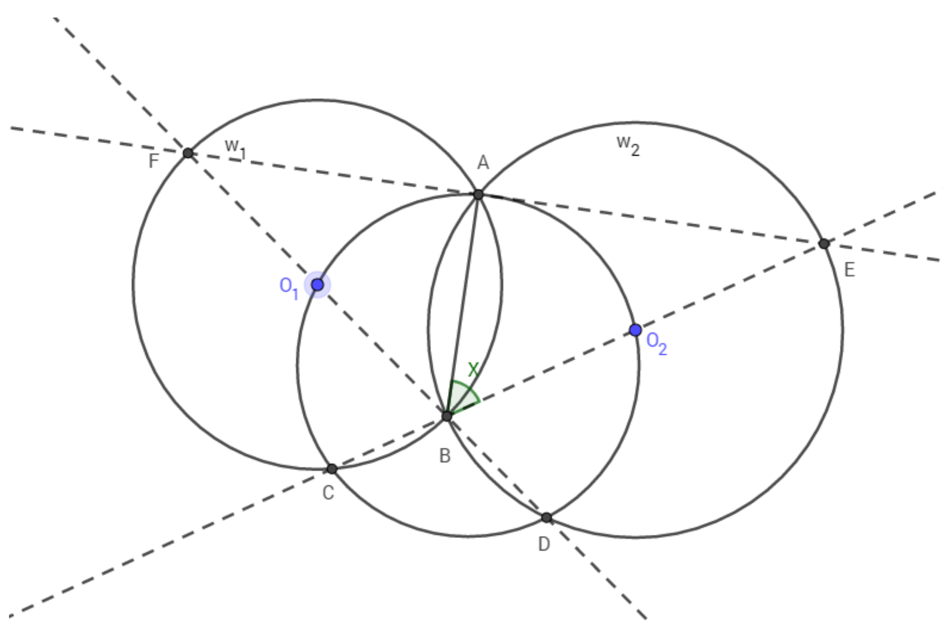
$$t_{2n+1} = 2t_{n-1}t_n + 2t_n^2 = t_n(2t_{n-1} + 2t_n)$$

which implies that t_n divides t_{2n+1}.

3. Let Γ_1 and Γ_2 be two circles with respective centres O_1 and O_2 intersecting in two distinct points A and B such that $\angle O_1AO_2$ is an obtuse angle. Let the circumcircle of triangle O_1AO_2 intersect Γ_1 and Γ_2 respectively in points C and D. Let the line CB intersect Γ_2 in E; let the line DB intersect Γ_1 in F. Prove that the points C, D, E, F are concyclic.

Solution: We will first prove that C, B, O_2, E are collinear; and this line is the bisector of $\angle ACD$:
Let $\angle ABO_2 = x$. Then by angle-chasing based on the given circles, we get

$$\angle AO_2B = (180 - 2x).$$



Hence $\angle AO_2O_1 = (90 - x)$. Since A, O_1, C, O_2 are concyclic, we obtain $\angle AO_2O_1 = \angle ACO_1 = (90 - x)$. Therefore $\angle AO_1C = 2x$. From this, we get $\angle AFC = x$ and $\angle ABC = 180 - x$. Thus, $\angle ABC$ and $\angle ABO_2$ are supplementary, implying C, B, O_2, E are collinear. Finally, we note that $O_2A = O_2D$ implies that O_2 is the midpoint of arc AO_2D; hence CO_2 is the bisector of $\angle ACD$, as required.

Similarly we obtain that D, B, O_1, F are collinear.
Hence, BE and BF are diameters of the respective circles. This shows that $\angle BAE = \angle BAF = 90°$; and hence F, A, E are collinear.
Finally, using all the above properties, we get:

$$\angle ECD = \angle BCD = \angle ACB = \angle AFB = \angle EFD.$$

Therefore C, D, E, F are concyclic, as required.

4. Find all polynomials with real coefficients $P(x)$ such that $P(x^2+x+1)$ divides $P(x^3-1)$.

Solution: We show that $P(x) = ax^n$ for some real number a and positive integer n. We prove that the only root of $P(x) = 0$ is 0. Suppose there is a root α_1 with $|\alpha_1| > 0$. Let β_1 and β_2 be the roots of $x^2 + x + 1 = \alpha_1$. Then $\beta_1 + \beta_2 = -1$. The given hypothesis shows that

$$P(\beta_1^3 - 1) = 0, \quad P(\beta_2^3 - 1) = 0.$$

We also see that

$$\beta_1^3 - 1 + \beta_2^3 - 1 = \alpha_1(\beta_1 + \beta_2 - 2).$$

Thus we have

$$|\beta_1^3 - 1| + |\beta_2^3 - 1| \geq |\beta_1^3 - 1 + \beta_2^3 - 1| = |\alpha_1||\beta_1 + \beta_2 - 2| = 3|\alpha_1|.$$

This shows that the absolute value of at least one of $\beta_1^3 - 1$ and $\beta_2^3 - 1$ is not less than $3|\alpha_1|/2$. If we take this as α_2, we have

$$|\alpha_2| > |\alpha_1|.$$

Now α_2 is a root of $P(x) = 0$ and we repeat the argument with α_2 in place of α_1. We get an infinite sequence of distinct roots of $P(x) = 0$. This contradiction proves $P(x) = ax^n$.

5. There are $n \geq 3$ girls in a class sitting around a circular table, each having some apples with her. Every time the teacher notices a girl having more apples than both of her neighbors combined, the teacher takes away one apple from that girl and gives one apple each to her neighbors. Prove that this process stops after a finite number of steps. (Assume that the teacher has an abundant supply of apples.)

Solution: Let $a_1, a_2, \ldots, a_n$ denote the number of apples with these girls at any given time, all taken in a circular way. Consider two quantities associated with this distribution: $s = a_1 + a_2 + \cdots + a_n$ and $t = a_1^2 + a_2^2 + \cdots + a_n^2$. Using Cauchy-Schwarz inequality, we see that

$$nt = n(a_1^2 + a_2^2 + \cdots + a_n^2) \geq (a_1 + a_2 + \cdots + a_n)^2 = s^2.$$

Therefore $t \geq s^2/n$ at any stage of the above process. Whenever teacher makes a move, s increases by 1. Suppose the girl with a_j apples has more than the sum of her neighbors. Then the change in t equals

$$(a_j-1)^2+(a_{j-1}+1)^2+(a_{j+1}+1)^2-a_j^2-a_{j-1}^2-a_{j+1}^2 = 2(a_{j+1}+a_{j-1}-a_j)+3 \leq 3+2(-1) = 1.$$

If s_1 and t_1 denote the corresponding sums after one move, we see that

$$s_1 = s + 1, \quad t_1 \le t + 1.$$

Thus after teacher performs k moves, if the corresponding sums are t_k and s_k, we obtain

$$t + k \ge t_k \ge \frac{s_k^2}{n} = \frac{(s+k)^2}{n}.$$

This leads to a quadratic inequality in k:

$$k^2 + k(2s - n) + (s^2 - nt) \le 0.$$

Since this cannot hold for large k, we see that the process must stop at some stage.

6. Let $\mathbb{N}$ denote the set of all natural numbers and let $f : \mathbb{N} \to \mathbb{N}$ be a function such that

(a) $f(mn) = f(m)f(n)$ for all m, n in $\mathbb{N}$;

(b) $m + n$ divides $f(m) + f(n)$ for all m, n in $\mathbb{N}$.

Prove that there exists an odd natural number k such that $f(n) = n^k$ for all n in $\mathbb{N}$.

Solution: Taking $m = n = 1$ in (a), we get $f(1) = 1$. Observe $f(2n) = f(2)f(n)$. Hence $2n+1$ divides $f(2n)+f(1) = f(2)f(n)+1$. This shows that $\gcd(f(2), 2n+1) = 1$ for all n. This means $f(2) = 2^k$ for some natural number k. Since $3 = 1 + 2$ divides $f(1)+f(2) = 1+2^k$, k is odd. Now take any arbitray power of 2, say 2^m, and an arbitray integer n. By (b), 2^m+n divides $f(2^m)+f(n)$. But (a) gives $f(2^m) = \big(f(2)\big)^m = 2^{km}$. Thus $2^m + n$ divides $2^{km} + f(n)$. But

$$2^{km} + f(n) = \big(2^{km} + n^k\big) + \big(f(n) - n^k\big) = M\big(2^m + n\big) + \big(f(n) - n^k\big),$$

since k is odd. It follows that $2^m + n$ divides $f(n) - n^k$. By Varying m over $\mathbb{N}$, we conclude that $f(n) - n^k = 0$. Therefore $f(n) = n^k$.

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