PRMO 2017

1. How many positive integers less than 1000 have the property that the sum of the digits of each such number is divisible by 7 and the number itself is divisible by 3?

2. Suppose a, b are positive real numbers such that $a\sqrt{a} + b\sqrt{b} = 183, \quad a\sqrt{b} + b\sqrt{a} = 182$. Find $\frac{9}{5}(a + b)$.

3. A contractor has two teams of workers: team A and team B. Team A can complete a job in 12 days and team B can do the same job in 36 days. Team A starts working on the job and team B joins team A after four days. The team A withdraws after two more days. For how many more days should team B work to complete the job?

4. Let a, b be integers such that all the roots of the equation $(x^2 + ax + 20)(x^2 + 17x + b) = 0$ are negative integers. What is the smallest possible value of $a + b$?

5. Let u, v, w be real numbers in geometric progression such that $u > v > w$. Suppose $u^{40} = v^n = w^{60}$. Find the value of n.

6. Let the sum $\sum_{n=1}^{9} \frac{1}{n(n+1)(n+2)}$ written in its lowest terms be $\frac{p}{q}$. Find the value of $q - p$.

7. Find the number of positive integers n, such that $\sqrt{n} + \sqrt{n+1} < 11$.

8 A pen costs ₹11 and a notebook costs ₹13. Find the number of ways in which a person can spend exactly ₹1000 to buy pens and notebooks.

9. There are five cities A, B, C, D, E on a certain island. Each city is connected to every other city by road. In how many ways can a person starting from city A come back to A after visiting some cities without visiting a city more than once and without taking the same road more than once? (The order in which he visits the cities also matters: e.g., the routes $A \to B \to C \to A$ and $A \to C \to B \to A$ are different.)

10. There are eight rooms on the first floor of a hotel, with four rooms on each side of the corridor, symmetrically situated (that is each room is exactly opposite to one other room). Four guests have to be accommodated in four of the eight rooms (that is, one in each) such that no two guests are in adjacent rooms or in opposite rooms. In how many ways can the guests be accommodated?

11. Let $f(x) = \sin \frac{x}{3} + \cos \frac{3x}{10}$ for all real x. Find the least natural number n such that $f(n\pi + x) = f(x)$ for all real x.

12. In a class, the total numbers of boys and girls are in the ratio $4 : 3$. On one day it was found that 8 boys and 14 girls were absent from the class, and that the number of boys was the square of the number of girls. What is the total number of students in the class?

13. In a rectangle $ABCD$, E is the midpoint of AB; F is a point on AC such that BF is perpendicular to AC; and FE perpendicular to BD. Suppose $BC = 8\sqrt{3}$. Find AB.

14. Suppose x is a positive real number such that $\{x\}, [x]$ and x are in a geometric progression. Find the least positive integer n such that $x^n > 100$. (Here $[x]$ denotes the integer part of x and $\{x\} = x - [x]$.)

15. Integers $1, 2, 3, \ldots, n$, where $n > 2$, are written on a board. Two numbers m, k such

that $1 < m < n$, $1 < k < n$ are removed and the average of the remaining numbers is found to be 17. What is the maximum sum of the two removed numbers?

16. Five distinct 2-digit numbers are in a geometric progression. Find the middle term.

17. Suppose the altitudes of a triangle are 10, 12 and 15. What is its semi-perimeter?

18. If the real numbers x, y, z are such that $x^2 + 4y^2 + 16z^2 = 48$ and $xy + 4yz + 2zx = 24$, what is the value of $x^2 + y^2 + z^2$?

19. Suppose $1, 2, 3$ are the roots of the equation $x^4 + ax^2 + bx = c$. Find the value of c.

20. What is the number of triples (a, b, c) of positive integers such that (i) $a < b < c < 10$ and (ii) $a, b, c, 10$ form the sides of a quadrilateral?

21. Find the number of ordered triples (a, b, c) of positive integers such that $abc = 108$.

22. Suppose in the plane 10 pairwise nonparallel lines intersect one another. What is the maximum possible number of polygons (with finite areas) that can be formed?

23. Suppose an integer x, a natural number n and a prime number p satisfy the equation $7x^2 - 44x + 12 = p^n$. Find the largest value of p.

24. Let P be an interior point of a triangle ABC whose sidelengths are 26, 65, 78. The line through P parallel to BC meets AB in K and AC in L. The line through P parallel to CA meets BC in M and BA in N. The line through P parallel to AB meets CA in S and CB in T. If KL, MN, ST are of equal lengths, find this common length.

25. Let $ABCD$ be a rectangle and let E and F be points on CD and BC respectively such that area$(ADE) = 16$, area$(CEF) = 9$ and area$(ABF) = 25$. What is the area of triangle AEF?

26. Let AB and CD be two parallel chords in a circle with radius 5 such that the centre O lies between these chords. Suppose $AB = 6$, $CD = 8$. Suppose further that the area of the part of the circle lying between the chords AB and CD is $(m\pi + n)/k$, where m, n, k are positive integers with $\gcd(m, n, k) = 1$. What is the value of $m + n + k$?

27. Let Ω_1 be a circle with centre O and let AB be a diameter of Ω_1. Let P be a point on the segment OB different from O. Suppose another circle Ω_2 with centre P lies in the interior of Ω_1. Tangents are drawn from A and B to the circle Ω_2 intersecting Ω_1 again at A_1 and B_1 respectively such that A_1 and B_1 are on the opposite sides of AB. Given that $A_1B = 5$, $AB_1 = 15$ and $OP = 10$, find the radius of Ω_1.

28. Let p, q be prime numbers such that $n^{3pq} - n$ is a multiple of $3pq$ for **all** positive integers n. Find the least possible value of $p + q$.

29. For each positive integer n, consider the highest common factor h_n of the two numbers $n! + 1$ and $(n + 1)!$. For $n < 100$, find the largest value of h_n.

30. Consider the areas of the four triangles obtained by drawing the diagonals AC and BD of a trapezium $ABCD$. The product of these areas, taken two at time, are computed. If among the six products so obtained, two products are 1296 and 576, determine the square root of the maximum possible area of the trapezium to the nearest integer.