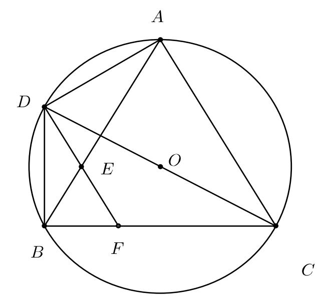
1. Let ABC be an isosceles triangle with $AB = AC$. Let Γ be its circumcircle and let O be the centre of Γ. Let CO meet Γ in D. Draw a line parallel to AC through D. Let it intersect AB at E. Suppose $AE : EB = 2 : 1$. Prove that ABC is an equilateral triangle.

   Solution: Extend DE to meet BC at F. Join BD and DA. Since CD is a diameter, we see that $\angle DBC = 90°$. Since DF is parallel to AC, it follows that $\triangle EBF \sim \triangle ABC$. Hence $EB = EF$. Now $\angle EBF = \angle EFB = 90° - \angle EDB$. But $\angle EBF + \angle EBD = 90°$. Hence we obtain $\angle EBD = \angle EDB$, which gives $EB = ED$. Since $AE : EB = 2 : 1$ and $EB = ED$, we obtain $AE = 2ED$. Hence $\angle DAB = 30°$. This implies $\angle DCB = 30°$ and hence $\angle BDC = 60°$. But then $\angle BAC = \angle BDC = 60°$ and hence $\triangle ABC$ is equilateral

2. Let a, b, c be positive real numbers such that

$$\frac{ab}{1+bc} + \frac{bc}{1+ca} + \frac{ca}{1+ab} = 1.$$

   Prove that $\dfrac{1}{a^3} + \dfrac{1}{b^3} + \dfrac{1}{c^3} \geq 6\sqrt{2}$.

   Solution: The given condition is equivalent to

$$\sum ab(1+ca)(1+ab) = (1+ab)(1+bc)(1+ca).$$

   This gives

$$\sum ab + \sum a^2b^2 + abc\sum a + abc\sum a^2 b = 1 + \sum ab + abc\sum a + a^2b^2c^2.$$

   Hence

$$a^2b^2c^2 + 1 = \sum a^2b^2 + abc\sum a^2b.$$

   Using

$$\sum a^2b^2 \geq 3(abc)^{4/3}, \quad \sum a^2 b \geq 3abc,$$

   we get

$$a^2b^2c^2 + 1 \geq 3(abc)^{4/3} + 3(abc)^2.$$

   Taking $x = (abc)^{2/3}$, this reduces to $2x^3 + 3x^2 - 1 \leq 0$. This gives $(x+1)^2(2x-1) \leq 0$. Hence $x \leq 1/2$. Therefore $abc \leq 1/2\sqrt{2}$. Finally

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} \geq \frac{3}{abc} \geq 6\sqrt{2}.$$

3. The present ages in years of two brothers A and B, and their father C are three distinct positive integers a, b, and c respectively. Suppose $\dfrac{b-1}{a-1}$ and $\dfrac{b+1}{a+1}$ are two consecutive integers, and $\dfrac{c-1}{b-1}$ and $\dfrac{c+1}{b+1}$ are two consecutive integers. If $a + b + c \leq 150$ determine a, b and c.

   Solution: We have

$$\frac{b-1}{a-1} = l, \quad \frac{b+1}{a+1} = l - 1, \quad \frac{c-1}{b-1} = m, \quad \frac{c+1}{b+1} = m - 1$$

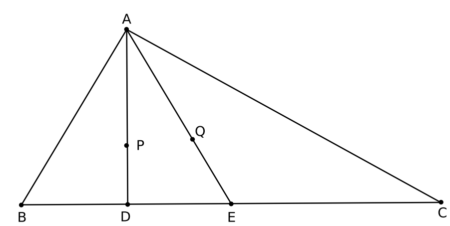
(If we take $a \leq b$, we see that $\frac{b-1}{a-1} \geq \frac{b+1}{a+1}$.)
Consider the first two relations: $b - 1 = l(a-1)$, $b + 1 = (l-1)(a+1)$. Solving for a, we get $a = 2l - 3$ and hence $b = 2l^2 - 4l + 1$. Using the second set of relations, we obtain $b = 2m - 3$ and $c = 2m^2 - 4m + 1$. Thius we have $2m - 3 = 2l^2 - 4l + 1$ or $m = (l-1)^2 + 1$. Obviously $l > 1$. If $l = 2$, we get $a = 1$ which forces $b = 1$ and $c = 1$, which is impossible. If $l = 3$, we get $a = 3$, $b = 7$ and $c = 31$. If $l \geq 4$, then $m \geq 10$ and $c \geq 161$. But then $a + b + c > 150$. Thus the only choice is $a = 3$, $b = 7$ and $c = 31$.

4. A box contains answer 4032 scripts out of which exactly half have odd number of marks. We choose 2 scripts randomly and, if the scores on both of them are odd number, we add one mark to one of them, put the script back in the box and keep the other script outside. If both scripts have even scores, we put back one of the scripts and keep the other outside. If there is one script with even score and the other with odd score, we put back the script with the odd score and keep the other script outside. After following this procedure a number of times, there are 3 scripts left among which there is at least one script each with odd and even scores. Find, with proof, the number of scripts with odd scores among the three left.

   Solution: There are three types of processes. In the first type, the scripts with odd scores decreases by 2. In the second and third types, there is no change in the number of scripts with odd scores. Hence at each step, the number of scripts with odd score decreases by 0 or 2. Since there are 2016 scripts with odd scores, the number of scripts with odd scores at the end is either 0 or 2. Since it is given that there is at least one script with odd scores, two of the three must have odd scores.

5. Let ABC be a triangle, AD an altitude and AE a median. Assume B, D, E, C lie in that order on the line BC. Suppose the incentre of triangle ABE lies on AD and the incentre of ADC lies on AE. Find the angles of triangle ABC.

   Solution: Since $AD \perp BE$ and the incentre of $\triangle ABE$ lies on AD, it follows that ABE is isosceles. In particular $\angle BAD = \angle DAE = \alpha$, say. Since AE is the bisector of$\angle DAC$, it follows that $\angle EAC = \angle DAE = \alpha$. Moreover, we have
$$\frac{AD}{AC} = \frac{DE}{CE}.$$
   Since $BE = EC = \frac{a}{2}$, we also have $DE = \frac{1}{2}BE = \frac{a}{4}$. Thus we get



$$\frac{AD}{AC} = \frac{DE}{EC} = \frac{a/4}{a/2} = \frac{1}{2}.$$
   Since $\triangle ADE$ is a right-angled triangle and $AD = AC/2$, it follows that $\angle ACD = 30°$. Hence $\angle DAC = 60°$. Since $\angle DAC = 2\alpha$, we get $\alpha = 30°$. Now $\angle A = 3\alpha$ and hence $\angle A = 90°$. This gives $\angle B = 60°$.

6. i) Prove that if an infinite sequence of strictly increasing positive integers in arithmetic progression has one cube then it has infinitely many cubes.
   (ii) Find, with justification, an infinite sequence of strictly increasing positive integers in arithmetic progression which does not have any cube.

   Solution:
   i) Let a be the first term of the AP and d be the common difference. (Here a, d are positive integers.) We can find an integer b such that $b^3 = a + (n-1)d$ for some $n \in \mathbb{N}$. Consider $(b+d)^3$. We observe that
$$(b+d)^3 = b^3 + d(3b^2 + 3bd + d^2) = a + (n - 1 + 3b^2 + 3bd + d^2)d = a + (m-1)d,$$

where $m = n + 3b^2 + 3bd + d^2$ is a positive integer. Hence $(b+d)^3$ is also in the given AP. More generally, the same method shows that $(b+kd)^3$ is in the AP for every $k \in \mathbb{N}$. Hence the given AP contains infinitely many cubes.

ii) Consider the AP $\langle 2, 6, 10, 14, \ldots \rangle$. Here $a = 2$ and $d = 4$. The general term is $2 + 4k$, where $k \geq 0$ is an integer. Suppose $2 + 4k = b^3$ for some integer b. Then 2 divides b. Hence $b = 2c$ for some c. Therefore $8c^3 = 2 + 4k$ or $4c^3 = 2k + 1$. But this is impossible since LHS is even and RHS is odd. We conclude that the AP $\langle 2, 6, 10, 14, \ldots \rangle$ does not contain any cube.

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