Regional Mathematical Olympiad 2015 (Mumbai region)
06 December, 2015

Hints and Solutions

1. Let $ABCD$ be a convex quadrilateral with $AB = a$, $BC = b$, $CD = c$ and $DA = d$. Suppose

$$a^2 + b^2 + c^2 + d^2 = ab + bc + cd + da,$$

and the area of $ABCD$ is 60 square units. If the length of one of the diagonals is 30 units, determine the length of the other diagonal.

Solution

$a^2+b^2+c^2+d^2 = ab+bc+cd+da \Rightarrow (a-b)^2+(b-c)^2+(c-d)^2+(d-a)^2 = 0 \Rightarrow a = b = c = d$. Thus $ABCD$ is a rhombus and

$$[ABCD] = (1/2)(d_1 d_2) \tag{1}$$

where d_1 and d_2 are the lengths of the diagonals. Hence $d_2 = \dfrac{2[ABCD]}{d_1} = 4$ units.

2. Determine the number of 3-digit numbers in base 10 having at least one 5 and at most one 3.

Solution

We count the number of 3-digit numbers with (i) at least one 5 and having no 3 and (ii) at least one 5 and having exactly one 3 separately.

(i) Here we first count the whole set and subtract the number of 3-digit numbers having no 5 from it. Since 3 is not there and 0 cannot be the first digit, we can fill the first digit in 8 ways. But we can fill the second and third digits in 9 ways(as 0 can be included). Thus we get $8 \times 9 \times 9$ such numbers. If no 5 is there, then the number of such numbers is $7 \times 8 \times 8$. Thus the number of 3-digit numbers not containing 3 and having at least one 5 is $(8 \times 9 \times 9) - (7 \times 8 \times 8) = 8(81 - 56) = 200$.

(ii) If 3 is there as a digit, then it can be the first digit or may be the second or third digit. Consider those numbers in which 3 is the first digit. The number of such numbers having at least one 5 is $(9 \times 9) - (8 \times 8) = 81 - 64 = 17$. The number of 3-digit numbers in which the second digit is 3 and having at least one 5 is $(8 \times 9) - (7 \times 8) = 16$. Similarly, the number of 3-digit numbers in which the third digit is 3 and having at least one 5 is $(8 \times 9) - (7 \times 8) = 16$. Thus we get $17 + 16 + 16 = 49$ such numbers.

Therefore the number of 3-digit numbers having at most one 3 and at least one 5 is $200+49 = 249$.

3. Let $P(x)$ be a non-constatnt polynomial whose coefficients are positive integers. If $P(n)$ divides $P(P(n) - 2015)$ for every natural number n, prove that $P(-2015) = 0$.

Solution

Note that $P(n) - 2015 - (-2015) = P(n)$ divides $P(P(n) - 2015) - P(-2015)$ for every positive integer n. But $P(n)$ divides $P(P(n) - 2015)$ for every positive integer n. Therefore $P(n)$ divides $P(-2015)$ for every positive integer n. Hence $P(-2015) = 0$.

Note

In the original version of the problem the word 'non-constant' was missing. The falsity of the statement was brought to the attention of the examiners by a contestant who mentioned it in a remark at the end of a perfect solution to the problem assuming that the polynomial is non-constant. Many students assumed that the polynomial is non-constant and completed the solution. They deserved full credit for doing so.

4. Find all three digit natural numbers of the form $(abc)_{10}$ such that $(abc)_{10}$, $(bca)_{10}$ and $(cab)_{10}$ are in geometric progression. (Here $(abc)_{10}$ is representation in base 10.)

Solution

Let us write

$$x = (a \times 10^2) + (b \times 10) + c, \quad y = (b \times 10^2) + (c \times 10) + a, \quad z = (c \times 10^2) + (a \times 10) + b.$$

We are given that $y^2 = xz$. This means

$$\big((b \times 10^2) + (c \times 10) + a\big)^2 = \big((a \times 10^2) + (b \times 10) + c\big)\big((c \times 10^2) + (a \times 10) + b\big).$$

We can solve for c and get

$$c = \frac{10b^2 - a^2}{10a - b}.$$

If a, b, c are digits leading to a solution, and if $d = \gcd(a, b)$ then $d|c$. Consequently, we may assume that $\gcd(a, b) = 1$. Now

$$c = \frac{999a^2}{10a - b} - (10b + 100a),$$

showing that $10a - b$ divides $999a^2$. Since a, b are relatively prime, this is possible only if $10a - b$ is a factor of 999. It follows that $10a - b$ takes the values 1,3,9,27,37. These values lead to the pairs

$$(a, b) = (1, 9), (1, 7), (1, 1), (4, 3).$$

We can discard the first two pairs as they lead to a value of $c > 10$. The third gives the trivial solution $(111, 111, 111)$. Taking $d = 2, 3, 4, 5, 6, 7, 8, 9$, we get 9 solution:

$$(abc)_{10} = 111, 222, 333, 444, 555, 666, 777, 888, 999.$$

The last pair gives $c = 2$ and hence the solution $(432, 324, 243)$. Another solution is obtained on multiplying by 2: $(864, 648, 486)$.

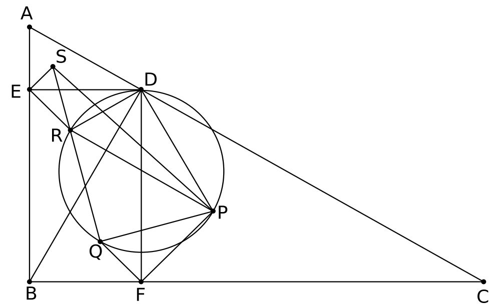
Thus we have

$$(abc)_{10} = 111, 222, 333, 444, 555, 666, 777, 888, 999, 432, 864.$$

5. Let ABC be a right-angled triangle with $\angle B = 90°$ and let BD be the altitude from B on to AC. Draw $DE \perp AB$ and $DF \perp BC$. Let P, Q, R and S be respectively the incentres of triangle DFC, DBF, DEB and DAE. Suppose S, R, Q are collinear. Prove that P, Q, R, D lie on a circle.

Solution

We first show that SR is perpendicular to QP. Consider triangles PFQ and RES. Observe that $AE \parallel DF$ and $ED \parallel FC$. Since ES bisects $\angle AED$ and FP bisects $\angle DFC$, it follows that $ES \parallel FP$. Since $ER \perp ES$ and $FQ \perp FP$, we also have $ER \parallel FQ$.



If r_1 and r_2 are inradii of triangles DEA and DEB, and if r' is the inradius of $\triangle DAB$, we know that

$$r_1 = r' \frac{AD}{AB}, \quad r_2 = r' \frac{BD}{AB}.$$

Hence

$$\frac{ES}{ER} = \frac{r_1}{r_2} = \frac{AD}{BD}.$$

Similarly, we can prove that

$$\frac{FQ}{FP} = \frac{BD}{DC}.$$

But we know that $AD/BD = BD/DC$. Hence we conclude that $ES/ER = FQ/FP$. Therefore $\triangle ESR \sim \triangle QFP$. But $SE \perp ER$ and $ER \parallel QF$ imply $SE \perp QF$. It follows that $SR \perp QP$.

Since S, R, Q are collinear, we get $SQ \perp QP$. Thus $\angle RQP = 90°$. Consider the circumcircle of $\triangle PRQ$. Since $\angle RDP = 90°$, we conclude that D lies on this circle. Hence P, Q, R, D are concyclic.

6. Let $S = \{1, 2, \ldots, n\}$ and let T be the set of all ordered triples of subsets of S, say (A_1, A_2, A_3), such that $A_1 \cup A_2 \cup A_3 = S$. Determine, in terms of n,

$$\sum_{(A_1,A_2,A_3)\in T} |A_1 \cap A_2 \cap A_3|$$

where $|X|$ denotes the number of elements in the set X. (For example, if $S = \{1, 2, 3\}$ and $A_1 = \{1, 2\}$, $A_2 = \{2, 3\}$, $A_3 = \{3\}$ then one of the elements of T is $(\{1, 2\}, \{2, 3\}, \{3\})$.)

Solution 1

Let $X = (A_1, A_2, A_3) \in T$ and let $i \in A_1 \cap A_2 \cap A_3$. The number of times the element i occurs in the required sum is equal to the number of ordered tuples $(A_1 - \{i\}, A_2 - \{i\}, A_3 - \{i\})$ such that

$$A_1 - \{i\} \cup A_2 - \{i\} \cup A_3 - \{i\} = S - \{i\} \qquad (*)$$

For every element of $S - \{i\}$, there are eight possibilities - whether the element belongs to or does not belong to A_i for $i = 1, 2, 3$. Out of these the case when the element does not belong to any of the three subsets violates $(*)$. Therefore, each element can satisfy the requirement in 7 ways. The number of tuples is, therefore, 7^{n-1} and the sum is $n.7^{n-1}$.

Solution 2

Consider all tuples (A_1, A_2, A_3) such that $A_1 \cap A_2 \cap A_3 = B \quad (*)$ and $A_1 \cup A_2 \cup A_3 = S \quad (**)$. Let $|B| = r$. (*) and (**) lead to

$$A_1 - B \cap A_2 - B \cap A_3 - B = \Phi \ \ (***) \text{ and } A_1 - B \cup A_2 - B \cup A_3 - B = S - B \ \ (****)$$

As before, for every element of $S - B$, there are eight possibilities. Out of these two cases -i. the element belongs to each of the three subsets A_1, A_2, A_3 violates (***) and ii. the element does not belong to any of the three subsets violates (****). Therefore, there are 6 possibilities for each element. Also, $|S - B| = n - r$, therefore, the number of tuples satisfying (*) and (**) is 6^{n-r}. The number of ways we can select r elements is $\binom{n}{r}$, therefore the required number is

$$\sum_{r=0}^{n} \binom{n}{r} r 6^{n-r} = n7^{n-1}$$

Solution 3

This solution was given by a contestant of standard IX. We provide a sketch of the same.

Write S_n and T_n in place of S and T respectively. Also, let

$$\sum_{(A_1,A_2,A_3)\in T_n} |A_1 \cap A_2 \cap A_3| = a_n.$$

We shall show that $a_n = n.7^{n-1}$ using recurrence and induction. Let $a_n = n.7^{n-1}$ for some positive integer n. Then, the elements of T_{n+1} are made by adding $n+1$ to either A_1 or A_2 or A_3 or $A_1 \cap A_2$ or $A_2 \cap A_3$ or $A_3 \cap A_1$ or $A_1 \cap A_2 \cap A_3$ in T_n. Further, it is easily established that $|T_n| = 7^n$. Now, if $n+1$ is added to $A_1 \cap A_2 \cap A_3$ then $|A_1 \cap A_2 \cap A_3|$ increases by 1. Otherwise it remains the same. So, as there are seven choices of where to put $n+1$,

$$a_{n+1} = 7a_n + |T_n|$$

Thus $a_{n+1} = (n+1).7^n$, which completes the inductive step, as the base case $n = 1$ is trivial. Therefore

$$\sum_{(A_1,A_2,A_3)\in T} |A_1 \cap A_2 \cap A_3| = n.7^{n-1}$$

7. Let x, y, z be real numbers such that $x^2 + y^2 + z^2 - 2xyz = 1$. Prove that

$$(1 + x)(1 + y)(1 + z) \leq 4 + 4xyz.$$

Solution

Write $1+2xyz = x^2+y^2+z^2 \Leftrightarrow 3+3xyz = \frac{3}{2}(x^2+y^2+z^2)+\frac{3}{2}$
$\Rightarrow 3+3xyz = (x^2+y^2+z^2)+\frac{1}{2}[(x^2+1)+(y^2+1)+(z^2+1)] \geq x^2+y^2+z^2+x+y+z$
(Use $x^2+y^2+z^2 \geq xy+yz+zx$ and AM-GM: $x^2+1 \geq 2x$ etc.)
$\Rightarrow 3+3xyz \geq xy+yz+zx+x+y+z$.
By adding $1+xyz$ in both sides,
we get $4+4xyz \geq 1+x+y+z+xy+yz+zx+xyz = (1+x)(1+y)(1+z)$.
Equality holds when $x=y=z=1$.

8. The length of each side of a convex quadrilateral $ABCD$ is a positive integer. If the sum of the lengths of any three sides is divisible by the length of the remaining side then prove that some two sides of the quadrilateral have the same length.

Solution

Let us take $ABCD$ to be a quadrilateral with $AB=a$, $BC=b$, $CD=c$ and $DA=d$ where a,b,c,d are integers. We know $AC < AB+BC$ and $AD < AC+CD < AB+BC+CD$. Thus we get $d < a+b+c$. Similarly we can write down three more inequalities: $a < b+c+d$, $b < c+d+a$ and $c < d+a+b$. We can also take d to be the largest side. Using the given conditions, we can write

$$a+b+c+d = la, \quad b+c+d+a = mb, \quad c+d+a+b = nc, \quad a+b+c+d = kd,$$

for some positive integers l,m,n,k.

Suppose no two sides are equal. Then $a<d$, $b<d$, $c<d$. Hence $a+b+c<3d$. Since d divides $a+b+c$ and $d<a+b+c<3d$, we must have $a+b+c=2d$. Thus we obtain

$$a+b+c+d = 3d = la = mb = nc.$$

Write this as

$$a = \frac{3d}{l}, \quad b = \frac{3d}{m}, \quad c = \frac{3d}{n}.$$

Using $2d = a+b+c$, we get the equation

$$\frac{3}{l}+\frac{3}{m}+\frac{3}{n} = 2.$$

Here l,m,n are necessarily distinct. Suppose $l=m$. Then $la=3d=lb$. This implies $a=b$, a contradiction. Similarly $m=n$ and $l=n$ can be discarded. We may assume $l<m<n$. This means, we have to solve the equation for distinct positive integers.

If $l \geq 4$, then $m \geq 5$ and $n \geq 6$. Hence

$$\frac{2}{3} = \frac{1}{l}+\frac{1}{m}+\frac{1}{n} \leq \frac{1}{4}+\frac{1}{5}+\frac{1}{6} = \frac{37}{60} < \frac{2}{3}$$

which is impossible. This means $l=1$, $l=2$ or $l=3$. For $l=1$, we get $b+c+d=0$, Which is impossible. If $l=2$, we get $b+c+d=a$, which contradicts $a<b+c+d$. If $l=3$, then $a=d$ and this contradicts again $a<d$. Therefore some two of a,b,c,d are equal.

THE END

NOTE

We do not claim that the solutions presented here are the most elegant solutions but we thought they would be instructive. For some problems we found that solutions by contestants were different and we have included them in this document.