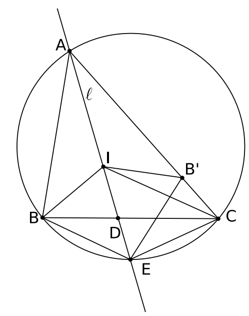
CRMO-2015 questions and solutions

1. Let ABC be a triangle. Let B' denote the reflection of B in the internal angle bisector ℓ of $\angle A$. Show that the circumcentre of the triangle $CB'I$ lies on the line ℓ, where I is the incentre of ABC.

   Solution: Let the line ℓ meet the circumcircle of ABC in E. Then E is the midpoint of the minor arc BC. Hence $EB = EC$.
   Note that $\angle EBC = \angle EAC = A/2$ and $\angle IBC = B/2$. Hence

   $$\angle BIE = \angle ABI + \angle BAI = B/2 + A/2.$$

   We also have

   $$\angle IBE = \angle IBC + \angle CBE = B/2 + A/2.$$

   Therefore $\angle BIE = \angle IBE$, so that $EB = EI$. Since AE is the perpendicular bisector of BB', we also have $EB = EB'$. Thus we get

   $$EB' = EC = EI.$$

   This implies that E is the circumcentre of $\triangle CB'I$

2. Let $P(x) = x^2 + ax + b$ be a quadratic polynomial where a is real and $b \neq 2$ is rational. Suppose $P(0)^2$, $P(1)^2$, $P(2)^2$ are integers. Prove that a and b are integers.

   Solution: We have $P(0) = b$. Since b is rational and $b^2 = P(0)^2$ is an integer, we conclude that b is an integer. Observe that

   $$\begin{aligned} P(1)^2 &= (1+a+b)^2 = a^2 + 2a(1+b) + (1+b)^2 \in \mathbb{Z} \\ P(2)^2 &= (4+2a+b)^2 = 4a^2 + 4a(4+b) + (4+b)^2 \in \mathbb{Z} \end{aligned}$$

   Eliminating a^2, we see that $4a(b-2) + 4(1+b)^2 - (4+b)^2 \in \mathbb{Z}$. Since $b \neq 2$, it follows that a is rational. Hence the equation $x^2 + 2x(1+b) + (1+b)^2 - \big(a^2 + 2a(1+b) + (1+b)^2\big) = 0$ is a quadratic equation with integer coefficients and has rational solution a. It follows that a is an integer.

3. Find all integers a, b, c such that

   $$a^2 = bc + 4, \quad b^2 = ca + 4.$$

   Solution: Suppose $a = b$. Then we get one equation: $a^2 = ac + 4$. This reduces to $a(a-c) = 4$. Therefore $a = 1$, $a - c = 4$; $a = -1$, $a - c = -4$; $a = 4, a - c = 1$; $a = -4, a - c = -1$; $a = 2, a - c = 2$; $a = -2, a - c = -2$. Thus we get $(a,b,c) = (1,1,-3)$, $(-1,-1,3)$, $(4,4,3)$, $(-4,-4,-3)$; $(2,2,0)$, $(-2,-2,0)$.
   If $a \neq b$, subtracting the second relation from the first we get

   $$a^2 - b^2 = c(b-a).$$

   This gives $a + b = -c$. Substituting this in the first equation, we get

   $$a^2 = b(-a-b) + 4.$$

Thus $a^2 + b^2 + ab = 4$. Multiplication by 2 gives

$$(a+b)^2 + a^2 + b^2 = 8.$$

Thus $(a, b) = (2, -2), (-2, 2), (2, 0), (-2, 0), (0, 2), (0, -2)$. We get respectively $c = 0, 0, -2, 2, -2, 2$. Thus we get the triples:

$$(a, b, c) = (1, 1, -3), (-1, -1, 3), (4, 4, 3), (-4, -4, -3), (2, 2, 0), (-2, -2, 0),$$
$$(2, -2, 0), (-2, 2, 0), (2, 0, -2), (-2, 0, 2), (0, 2, -2), (0, -2, 2).$$

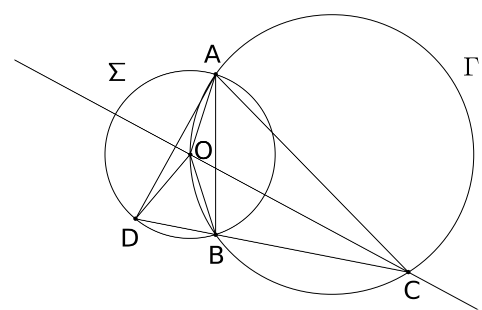
4. Suppose 40 objects are placed along a circle at equal distances. In how many ways can 3 objects be chosen from among them so that no two of the three chosen objects are adjacent nor diametrically opposite?

Solution: One can choose 3 objects out of 40 objects in $\binom{40}{3}$ ways. Among theese choices all would be together in 40 cases; exactly two will be together in 40×36 cases. Thus three objects can be chosen such that no two adjacent in $\binom{40}{3} - 40 - (40 \times 36)$ ways. Among these, furthrer, two objects will be diametrically opposite in 20 ways and the third would be on either semicircle in a non adjacent portion in $40 - 6 = 34$ ways. Thus required number is

$$\binom{40}{3} - 40 - (40 \times 36) - (20 \times 34) = 7720.$$

5. Two circles Γ and Σ intersect at two distinct points A and B. A line through B intersects Γ and Σ again at C and D, respectively. Suppose that $CA = CD$. Show that the centre of Σ lies on Γ.

Solution: Let the perpendicular from C to AD intersect Γ at O. Since $CA = CD$ we have that CO is the perpendicular bisector of AD and also the angular bisector of $\angle ACD$. From the former, it follows that $OA = OD$, and from the latter it follows that $\angle OCB = \angle OCA$ and hence $OA = OB$. Thus we get $OA = OB = OD$. This means O is the circumcentre of triangle ADB. This shows that O is the centre of Σ.



6. How many integers m satisfy both the following properties:

(i) $1 \le m \le 5000$; (ii) $\left[\sqrt{m}\right] = \left[\sqrt{m+125}\right]$?

(Here $[x]$ denotes the largest integer not exceeding x, for any real number x.)

Solution: Let $\left[\sqrt{m}\right] = \left[\sqrt{m+125}\right] = k$. Then we know that

$$k^2 \le m < m + 125 < (k+1)^2.$$

Thus

$$m + 125 < k^2 + 2k + 1 \le m + 2k + 1.$$

This shows that $2k + 1 > 125$ or $k > 62$. Using $k^2 \le 5000$, we get $k \le 70$. Thus $k \in \{63, 64, 65, 66, 67, 68, 69, 70\}$. We observe that $63^2 = 3969$ and $64^2 = 63^2 + 127$. Hence

$$\left[\sqrt{63^2 + 125}\right] = \left[\sqrt{63^2 + 1 + 125}\right] = 63,$$

but $\left[\sqrt{63^2+2+125}\right]=64$. Thus we get two values of m such that $\left[\sqrt{m}\right]=\left[\sqrt{m+125}\right]$ for $k=63$. Similarly, $65^2=64^2+129$ so that

$$\left[\sqrt{64^2+125}\right]=\left[\sqrt{64^2+1+125}\right]=\left[\sqrt{64^2+2+125}\right]=\left[\sqrt{64^2+3+125}\right]=64,$$

but $\left[\sqrt{64^2+4+125}\right]=65$. Thus we get four values of m such that $\left[\sqrt{m}\right]=\left[\sqrt{m+125}\right]$ for $k=64$. Continuing, we see that there are $6,8,10,12,14,16$ values of m respectively for $k=65,66,67,68,69,70$. Together we get

$$2+4+6+8+10+12+14+16=2\times\frac{8\times 9}{2}=72$$

values of m satisfying the given requirement.

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