Solutions to RMO-2014 problems

1. Let $ABCD$ be an isosceles trapezium having an incircle; let AB and CD be the parallel sides and let CE be the perpendicular from C on to AB. Prove that CE is equal to the geometric mean of AB and CD.

   Solution: Since $ABCD$ has incircle, we have $AB + CD = AD + BC$. We alos know that $AD = BC$ and $\angle A = \angle B$. Draw $DF \perp AB$. Then $\triangle DFC \cong \triangle CEB$. Hence $FE = CD$, and $AF = EB$. Now

   $$CE^2 = BC^2 - BE^2.$$

   Observe $2BC = BC + AD = AB + CD = 2FE + 2EB$. Hence $BC = FE + EB$. Thus

   $$CE^2 = (FE + EB)^2 - BE^2 = (FE + 2EB)FE = AB \cdot CD.$$

   This shows that CE is the geometric mean of AB and CD.

2. If x and y are positive real numbers, prove that

   $$4x^4 + 4y^3 + 5x^2 + y + 1 \geq 12xy.$$

   Solution: We have from AM-GM inequality,

   $$4x^4 + 1 \geq 4x^2, \quad 4y^3 + y = y(4y^2 + 1) \geq 4y^2.$$

   Hence

   $$\begin{aligned} 4x^4 + 4y^3 + 5x^2 + y + 1 &\geq 4x^2 + 4y^2 + 5x^2 \\ &= 9x^2 + 4y^2 \\ &\geq 2(\sqrt{9 \times 4})xy \\ &= 12xy. \end{aligned}$$

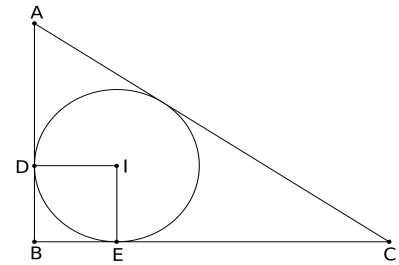
3. Determine all pairs $m > n$ of positive integers such that

   $$1 = \gcd(n+1, m+1) = \gcd(n+2, m+2) = \cdots = \gcd(m, 2m-n).$$

   Solution: Observe that $1 = \gcd(n+r, m+r) = \gcd(n+r, m-n)$. Thus each of the $m - n$ consecutive positive integers $n+1, n+2, \ldots, m$ is coprime to $m - n$. Since one of these is necessarily a multiple of $m - n$, this is possible only when $m - n = 1$. Hence each pair is of the form $(n, n+1)$, where $n \in \mathbb{N}$.

4. What is the minimal area of a right-angled triangle whose inradius is 1 unit?

Solution: Let ABC be the right-angled triangle with $\angle B = 90°$. Let I be its incentre and D be the point where the incircle touches AB. Then $s-b = AD = r = 1$. We also know that $[ABC] = rs = r(a+b+c)/2$ and $[ABC] = ac/2$. Thus



$$\frac{ac}{2} = \frac{a+b+c}{2} = (a+c) - \frac{a+c-b}{2} = (a+c)-1.$$

Using AM-GM inequality, we get

$$\frac{ac}{2} = a+c-1 \geq 2\sqrt{ac}-1.$$

Taking $\sqrt{ac} = x$, we get $x^2 - 4x + 2 \geq 0$. Hence

$$x \geq \frac{4+2\sqrt{2}}{2} = 2+\sqrt{2}.$$

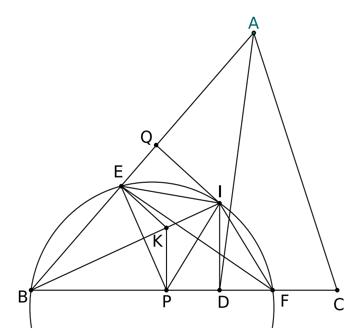
Finally,

$$[ABC] = \frac{ac}{2} \geq \frac{(2+\sqrt{2})^2}{2} = 3+2\sqrt{2}.$$

Thus the least area of such a triangle is $3+2\sqrt{2}$.

5. Let ABC be an acute-angled triangle and let I be its incentre. Let the incircle of triangle ABC touch BC in D. The incircle of the triangle ABD touches AB in E; the incircle of the triangle ACD touches BC in F. Prove that B, E, I, F are concyclic.

Solution: We know $BD = s-b$ and $DC = s-c$, where s is the semiperimeter of $\triangle ABC$. Let the incircle of $\triangle ABD$ touch BC in P and let $AD = l$. Then

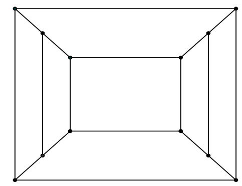


$$DP = \frac{l+BD-c}{2} = \frac{l+s-c-b}{2}.$$

Similarly, we can compute $DF = \dfrac{l+s-c-b}{2}$. Therefore $DP = DF$. But $ID \perp BC$. Hence I is on the perpendicular bisector of PF. This gives $IP = IF$.

Draw $IQ \perp AB$. Then B, Q, I, D are concyclic so that $\angle QID = 180° - \angle B$. Since $DP = DF$ and $IP = IF$, the triangles IDP and IDF are congruent. But IDP is congruent to IQE. It follows that $\triangle IDF \cong \triangle IQE$. This shows that $\angle QIE = \angle DIF$. Therefore $\angle QID = \angle EIF$. But $\angle QID = 180° - \angle B$. Hence $\angle EIF = 180° - \angle B$. Therefore B, E, I, F are concyclic.

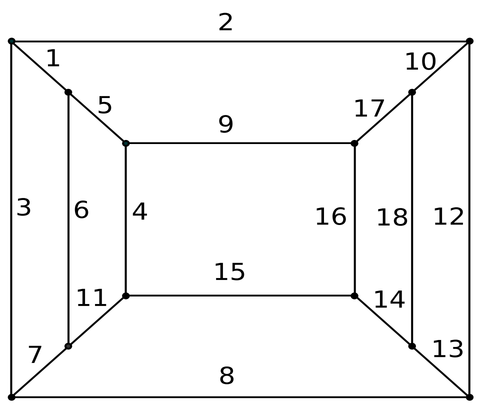
6. In the adjacent figure, can the numbers $1, 2, 3, 4, \cdots, 18$ be placed, one on each line segment, such that the sum of the numbers on the three line segments meeting at each point is divisible by 3?



Solution: We group the numbers 1 to 18 in to 3 groups: those leaving remainder 0 when divided by 3; those leaving remainder 1; and those leaving remainder 2. Thus the groups are:

$$\{3, 6, 9, 12, 15, 18\},\ \{1, 4, 7, 10, 13, 16\},\ \{2, 5, 8, 11, 14, 17\}$$

Now we put the numbers in such a way that each of the three line segments converging to a vertex gets one number from each set. For example, here is one such arrangement:



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