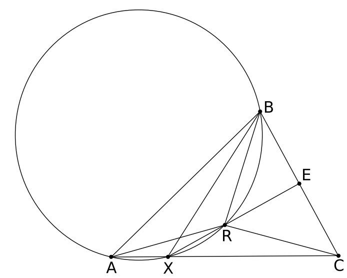
Solutions to RMO-2014 problems

1. Let ABC be an acute-angled triangle and suppose $\angle ABC$ is the largest angle of the triangle. Let R be its circumcentre. Suppose the circumcircle of triangle ARB cuts AC again in X. Prove that RX is pependicular to BC.

   Solution: Extend RX to meet BC in E. We show that $\angle XEC = 90°$. Join RA, RB and BX. Observe that $\angle AXB = \angle ARB = 2\angle C$ and $\angle BXR = \angle BAR = 90° - \angle C$. Hence $\angle EXC = 180° - 2\angle C - (90° - \angle C) = 90° - \angle C$. This shows that $\angle CEX = 90°$.



2. Find all real numbers x and y such that

$$x^2 + 2y^2 + \frac{1}{2} \leq x(2y+1).$$

   Solution: We write the inequality in the form

$$2x^2 + 4y^2 + 1 - 4xy - 2x \leq 0.$$

   Thus $(x^2 - 4xy + 4y^2) + (x^2 - 2x + 1) \leq 0$. Hence

$$(x-2y)^2 + (x-1)^2 \leq 0.$$

   Since x, y are real, we know that $(x-2y)^2 \geq 0$ and $(x-1)^2 \geq 0$. Hence it follows that $(x-2y)^2 = 0$ and $(x-1)^2 = 0$. Therefore $x = 1$ and $y = 1/2$.

3. Prove that there does not exist any positive integer $n < 2310$ such that $n(2310 - n)$ is a multiple of 2310.

   Solution: Suppose there exists n such that $0 < n < 2310$ and $n(2310-n) = 2310k$. Then $n^2 = 2310(n-k)$. But $2310 = 2 \times 3 \times 5 \times 7 \times 11$, the product of primes. Hence $n - k = 2310l^2$ for some l. But $n < 2310$ and hence $n - k < 2310$. Hence $l = 0$. This forces $n = k$ and hence $n^2 = 2310(n-k) = 0$. Thus $n = 0$ and we have a contradiction.

4. Find all positive real numbers x, y, z such that

$$2x - 2y + \frac{1}{z} = \frac{1}{2014}, \quad 2y - 2z + \frac{1}{x} = \frac{1}{2014}, \quad 2z - 2x + \frac{1}{y} = \frac{1}{2014}.$$

   Solution: Adding the three equations, we get

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{3}{2014}.$$

We can also write the equations in the form

$$2zx - 2zy + 1 = \frac{z}{2014}, \quad 2xy - 2xz + 1 = \frac{x}{2014}, \quad 2yz - 2yx + 1 = \frac{y}{2014}.$$

Adding these, we also get

$$2014 \times 3 = x + y + z.$$

Therefore

$$\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)(x + y + z) = \frac{3}{2014} \times (2014 \times 3) = 9.$$

Using AM-GM inequality, we therefore obtain

$$9 = \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)(x + y + z) \geq 9 \times (xyz)^{1/3}\left(\frac{1}{xyz}\right)^{1/3} = 9.$$

Hence equality holds in AM-GM inequality and we conclude $x = y = z$. Thus

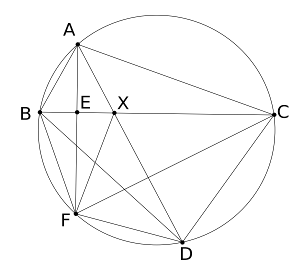
$$\frac{1}{x} = \frac{1}{2014}$$

which gives $x = 2014$. We conclude

$$x = 2014, \quad y = 2014, \quad z = 2014.$$

5. Let ABC be a triangle. Let X be on the segment BC such that $AB = AX$. Let AX meet the circumcircle Γ of triangle ABC again at D. Show that the circumcentre of $\triangle BDX$ lies on Γ.

Solution: Draw perpendicular from A to BC and extend it to meet Γ in F. We show that F is the circumcentre of $\triangle BDX$. Since $AB = AX$, we observe that F lies on the perpendicular bisector of BX. Join CF and CD. We observe that $\angle ABX = \angle CDX$ and $\angle AXB = \angle CXD$. Hence $\triangle ABX$ is similar to $\triangle CDX$. In particular $\triangle CDX$ is isosceles.



Moreover, $\angle BCF = \angle BAF$ and $\angle DCF = \angle DAF$. Since AF is the perpendicular bisector of BX, it also bisects $\angle BAX$. It follows that CF bisects $\angle DCX$ and hence F lies on the perpendicular bisector of DX. Together F is the circumcentre of $\triangle BXD$.

6. For any natural number n, let $S(n)$ denote the sum of the digits of n. Find the number of all 3-digit numbers n such that $S(S(n)) = 2$.

Solution: Observe that $S(S(n)) = 2$ implies that $S(n) = 2, 11$ or 20. Hence we have to find the number of all all 3 digit numbers $\overline{abc}$ such that $a + b + c = 2$, 11 and 20. In fact we can enumerate all these:

$$a+b+c=2\text{: } \overline{abc}=101,110,200;$$
$$a+b+c=11;\ \overline{abc}=902,920,290,209,911,191,119,803,830,308,380,$$
$$812,821,182,128,218,281,731,713,317,371,137,173,722,272,227,740,704,$$
$$407,470,650,605,560,506,641,614,416,461,164,146,623,632,362,326,263,236;$$
$$a+b+c=20;\ \overline{abc}=992,929,299,983,938,398,389,839,893,974,947,794,749,$$
$$479,497,965,956,659,695,596,569,884,848,488,$$
$$875,875,785,758,578,587,866,686,668,776,767,677.$$

There are totally 85 three digit numbers having second digital sum equal to 2.

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