NATIONAL BOARD FOR HIGHER MATHEMATICS

AND

HOMI BHABHA CENTRE FOR SCIENCE EDUCATION
TATA INSTITUTE OF FUNDAMENTAL RESEARCH

Pre-REGIONAL MATHEMATICAL OLYMPIAD, 2014
Mumbai Region

October 12, 2014

QUESTION PAPER SET: A

- There are 20 questions in this question paper. Each question carries 5 marks.

- Answer all questions.

- Time allotted: 2.5 hours.

QUESTIONS

1. A natural number k is such that $k^2 < 2014 < (k+1)^2$. What is the largest prime factor of k?

2. The first term of a sequence is 2014. Each succeeding term is the sum of the cubes of the digits of the previous term. What is the 2014th term of the sequence?

3. Let $ABCD$ be a convex quadrilateral with perpendicular diagonals. If $AB = 20, BC = 70$ and $CD = 90$, then what is the value of DA?

4. In a triangle with integer side lengths, one side is three times as long as a second side, and the length of the third side is 17. What is the greatest possible perimeter of the triangle?

5. If real numbers a, b, c, d, e satisfy

$$a + 1 = b + 2 = c + 3 = d + 4 = e + 5 = a + b + c + d + e + 3,$$

what is the value of $a^2 + b^2 + c^2 + d^2 + e^2$?

6. What is the smallest possible natural number n for which the equation $x^2 - nx + 2014 = 0$ has integer roots?

7. If $x^{(x^4)} = 4$, what is the value of $x^{(x^2)} + x^{(x^8)}$?

8. Let S be a set of real numbers with mean M. If the means of the sets $S \cup \{15\}$ and $S \cup \{15, 1\}$ are $M + 2$ and $M + 1$, respectively, then how many elements does S have?

9. Natural numbers k, l, p and q are such that if a and b are roots of $x^2 - kx + l = 0$ then $a + \frac{1}{b}$ and $b + \frac{1}{a}$ are the roots of $x^2 - px + q = 0$. What is the sum of all possible values of q?

10. In a triangle ABC, X and Y are points on the segments AB and AC, respectively, such that $AX : XB = 1 : 2$ and $AY : YC = 2 : 1$. If the area of triangle AXY is 10 then what is the area of triangle ABC?

11. For natural numbers x and y, let (x, y) denote the greatest common divisor of x and y. How many pairs of natural numbers x and y with $x \le y$ satisfy the equation $xy = x + y + (x, y)$?

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12. Let $ABCD$ be a convex quadrilateral with $\angle DAB = \angle BDC = 90°$. Let the incircles of triangles ABD and BCD touch BD at P and Q, respectively, with P lying in between B and Q. If $AD = 999$ and $PQ = 200$ then what is the sum of the radii of the incircles of triangles ABD and BDC?

13. For how many natural numbers n between 1 and 2014 (both inclusive) is $\dfrac{8n}{9999 - n}$ an integer?

14. One morning, each member of Manjul's family drank an 8-ounce mixture of coffee and milk. The amounts of coffee and milk varied from cup to cup, but were never zero. Manjul drank 1/7-th of the total amount of milk and 2/17-th of the total amount of coffee. How many people are there in Manjul's family?

15. Let XOY be a triangle with $\angle XOY = 90°$. Let M and N be the midpoints of legs OX and OY, respectively. Suppose that $XN = 19$ and $YM = 22$. What is XY?

16. In a triangle ABC, let I denote the incenter. Let the lines AI, BI and CI intersect the incircle at P, Q and R, respectively. If $\angle BAC = 40°$, what is the value of $\angle QPR$ in degrees?

17. For a natural number b, let $N(b)$ denote the number of natural numbers a for which the equation $x^2 + ax + b = 0$ has integer roots. What is the smallest value of b for which $N(b) = 20$?

18. Let f be a one-to-one function from the set of natural numbers to itself such that $f(mn) = f(m)f(n)$ for all natural numbers m and n. What is the least possible value of $f(999)$?

19. Let $x_1, x_2, \cdots, x_{2014}$ be real numbers different from 1, such that $x_1 + x_2 + \cdots + x_{2014} = 1$ and

$$\frac{x_1}{1 - x_1} + \frac{x_2}{1 - x_2} + \cdots + \frac{x_{2014}}{1 - x_{2014}} = 1.$$

What is the value of

$$\frac{x_1^2}{1 - x_1} + \frac{x_2^2}{1 - x_2} + \frac{x_3^2}{1 - x_3} + \cdots + \frac{x_{2014}^2}{1 - x_{2014}} \, ?$$

20. What is the number of ordered pairs (A, B) where A and B are subsets of $\{1, 2, \ldots, 5\}$ such that neither $A \subseteq B$ nor $B \subseteq A$?