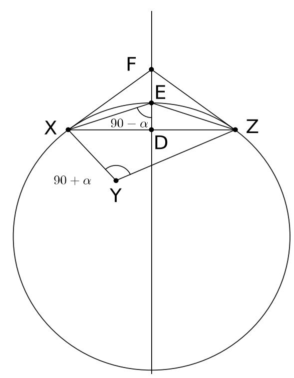
Problems and Solutions: INMO-2015

1. Let ABC be a right-angled triangle with $\angle B = 90°$. Let BD be the altitude from B on to AC. Let P, Q and I be the incentres of triangles ABD, CBD and ABC respectively. Show that the circumcentre of of the triangle PIQ lies on the hypotenuse AC.

   Solution: We begin with the following lemma:

   Lemma: Let XYZ be a triangle with $\angle XYZ = 90 + \alpha$. Construct an isosceles triangle XEZ, externally on the side XZ, with base angle α. Then E is the circumcentre of $\triangle XYZ$.

   Proof of the Lemma: Draw $ED \perp XZ$. Then DE is the perpendicular bisector of XZ. We also observe that $\angle XED = \angle ZED = 90 - \alpha$. Observe that E is on the perpendicular bisector of XZ. Construct the circumcircle of XYZ. Draw perpendicular bisector of XY and let it meet DE in F. Then F is the circumcentre of $\triangle XYZ$. Join XF. Then $\angle XFD = 90 - \alpha$. But we know that $\angle XED = 90 - \alpha$. Hence $E = F$.



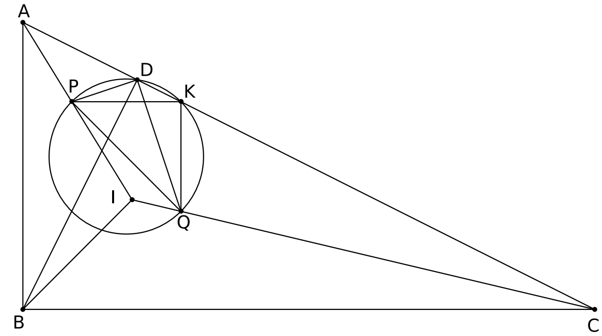
   Let r_1, r_2 and r be the inradii of the triangles ABD, CBD and ABC respectively. Join PD and DQ. Observe that $\angle PDQ = 90°$. Hence

$$PQ^2 = PD^2 + DQ^2 = 2r_1^2 + 2r_2^2.$$

   Let $s_1 = (AB + BD + DA)/2$. Observe that $BD = ca/b$ and $AD = \sqrt{AB^2 - BD^2} = \sqrt{c^2 - \left(\frac{ca}{b}\right)^2} = c^2/b$. This gives $s_1 = cs/b$. But $r_1 = s_1 - c = (c/b)(s - b) = cr/b$. Similarly, $r_2 = ar/b$. Hence

$$PQ^2 = 2r^2\left(\frac{c^2 + a^2}{b^2}\right) = 2r^2.$$

   Consider $\triangle PIQ$. Observe that $\angle PIQ = 90 + (B/2) = 135$. Hence PQ subtends $90°$ on the circumference of the circumcircle of $\triangle PIQ$. But we have seen that $\angle PDQ = 90°$. Now construct a circle with PQ as diameter. Let it cut AC again in K. It follows that $\angle PKQ = 90°$ and the points P, D, K, Q are concyclic. We also notice $\angle KPQ = \angle KDQ = 45°$ and $\angle PQK = \angle PDA = 45°$.

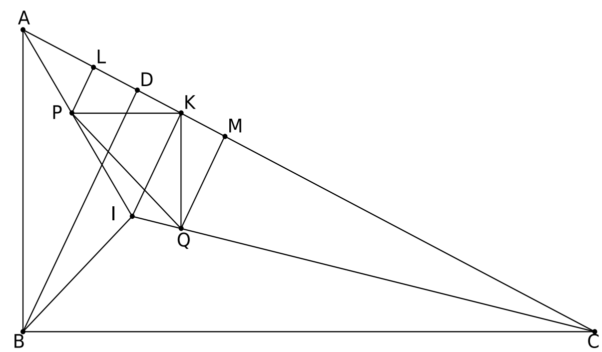


Thus PKQ is an isosceles right-angled triangle with $KP = KQ$. Therfore $KP^2 + KQ^2 = PQ^2 = 2r^2$ and hence $KP = KQ = r$.

Now $\angle PIQ = 90 + 45$ and $\angle PKQ = 2 \times 45° = 90°$ with $KP = KQ = r$. Hence K is the circumcentre of $\triangle PIQ$.

(Incidentally, This also shows that $KI = r$ and hence K is the point of contact of the incircle of $\triangle ABC$ with AC.)

Solution 2: Here we use computation to prove that the point of contact K of the incircle with AC is the circumcentre of $\triangle PIQ$. We show that $KP = KQ = r$. Let r_1 and r_2 be the inradii of triangles ABD and CBD respectively. Draw $PL \perp AC$ and $QM \perp AC$. If s_1 is the semiperimeter of $\triangle ABD$, then $AL = s_1 - BD$.



But
$$s_1 = \frac{AB + BD + DA}{2}, \quad BD = \frac{ca}{b}, \quad AD = \frac{c^2}{b}$$
Hence $s_1 = cs/b$. This gives $r_1 = s_1 - c = cr/b$, $AL = s_1 - BD = c(s-a)/b$. Hence $KL = AK - AL = (s-a) - \frac{c(s-a)}{b} = \frac{(b-c)(s-a)}{b}$. We observe that

$$2r^2 = \frac{(c+a-b)^2}{2} = \frac{c^2 + a^2 + b^2 - 2bc - 2ab + 2ca}{2} = (b^2 - ba - bc + ac) = (b-c)(b-a).$$

This gives

$$\begin{aligned}(s-a)(b-c) &= (s-b+b-a)(b-c) = r(b-c) + (b-a)(b-c)\\ &= r(b-c) + 2r^2 = r(b-c+c+a-b) = ra.\end{aligned}$$

Thus $KL = ra/b$. Finally,

$$KP^2 = KL^2 + LP^2 = \frac{r^2a^2}{b^2} + \frac{r^2 + c^2}{b^2} = r^2.$$

Thus $KP = r$. Similarly, $KQ = r$. This gives $KP = KI = KQ = r$ and therefore K is the circumcentre of $\triangle KIQ$.

(Incidentally, this also shows that $KL = ca/b = r_2$ and $KM = r_1$.)

2. For any natural number $n > 1$, write the infinite decimal expansion of $1/n$ (for example, we write $1/2 = 0.4\bar{9}$ as its infinite decimal expansion, not 0.5). Determine the length of the non-periodic part of the (infinite) decimal expansion of $1/n$.

Solution: For any prime p, let $\nu_p(n)$ be the maximum power of p dividing n; ie $p^{\nu_p(n)}$ divides n but not higher power. Let r be the length of the non-periodic part of the infinite decimal expansion of $1/n$.

Write

$$\frac{1}{n} = 0.a_1a_2\cdots a_r\overline{b_1b_2\cdots b_s}.$$

We show that $r = \max(\nu_2(n), \nu_5(n))$.

Let a and b be the numbers $a_1a_2\cdots a_r$ and $b = b_1b_2\cdots b_s$ respectively. (Here a_1 and b_1 can be both 0.) Then

$$\frac{1}{n} = \frac{1}{10^r}\left(a + \sum_{k\geq 1}\frac{b}{(10^s)^k}\right) = \frac{1}{10^r}\left(a + \frac{b}{10^s-1}\right).$$

Thus we get $10^r(10^s-1) = n\big((10^s-1)a+b\big)$. It shows that $r \geq \max(\nu_2(n), \nu_5(n))$. Suppose $r > \max(\nu_2(n), \nu_5(n))$. Then 10 divides $b-a$. Hence the last digits of a and b are equal: $a_r = b_s$. This means

$$\frac{1}{n} = 0.a_1a_2\cdots a_{r-1}\overline{b_sb_1b_2\cdots b_{s-1}}.$$

This contradicts the definition of r. Therefore $r = \max(\nu_2(n), \nu_5(n))$.

3. Find all real functions f from $\mathbb{R} \to \mathbb{R}$ satisfying the relation

$$f(x^2 + yf(x)) = xf(x+y).$$

Solution: Put $x = 0$ and we get $f\big(yf(0)\big) = 0$. If $f(0) \neq 0$, then $yf(0)$ takes all real values when y varies over real line. We get $f(x) \equiv 0$. Suppose $f(0) = 0$. Taking $y = -x$, we get $f\big(x^2 - xf(x)\big) = 0$ for all real x.

Suppose there exists $x_0 \neq 0$ in $\mathbb{R}$ such that $f(x_0) = 0$. Putting $x = x_0$ in the given relation we get

$$f\big(x_0^2\big) = x_0f(x_0+y),$$

for all $y \in \mathbb{R}$. Now the left side is a constant and hence it follows that f is a constant function. But the only constant function which satisfies the equation is identically zero function, which is already obtained. Hence we may consider the case where $f(x) \neq 0$ for all $x \neq 0$.

Since $f\big(x^2 - xf(x)\big) = 0$, we conclude that $x^2 - xf(x) = 0$ for all $x \neq 0$. This implies that $f(x) = x$ for all $x \neq 0$. Since $f(0) = 0$, we conclude that $f(x) = x$ for all $x \in R$.

Thus we have two functions: $f(x) \equiv 0$ and $f(x) = x$ for all $x \in \mathbb{R}$.

4. There are four basket-ball players A, B, C, D. Initially, the ball is with A. The ball is always passed from one person to a different person. In how many ways can the ball come back to A after **seven** passes? (For example $A \to C \to B \to D \to A \to B \to C \to A$ and

$A \to D \to A \to D \to C \to A \to B \to A$ are two ways in which the ball can come back to A after seven passes.)

Solution: Let x_n be the number of ways in which A can get back the ball after n passes. Let y_n be the number of ways in which the ball goes back to a fixed person other than A after n passes. Then

$$x_n = 3y_{n-1},$$

and

$$y_n = x_{n-1} + 2y_{n-1}.$$

We also have $x_1 = 0$, $x_2 = 3$, $y_1 = 1$ and $y_2 = 2$.

Eliminating y_n and y_{n-1}, we get $x_{n+1} = 3x_{n-1} + 2x_n$. Thus

$$\begin{aligned}
x_3 &= 3x_1 + 2x_2 = 2 \times 3 = 6; \\
x_4 &= 3x_2 + 2x_3 = (3 \times 3) + (2 \times 6) = 9 + 12 = 21; \\
x_5 &= 3x_3 + 2x_4 = (3 \times 6) + (2 \times 21) = 18 + 42 = 60; \\
x_6 &= 3x_4 + 2x_5 = (3 \times 21) + (2 \times 60) = 63 + 120 = 183; \\
x_7 &= 3x_5 + 2x_6 = (3 \times 60) + (2 \times 183) = 180 + 366 = 546.
\end{aligned}$$

Alternate solution: Since the ball goes back to one of the other 3 persons, we have

$$x_n + 3y_n = 3^n,$$

since there are 3^n ways of passing the ball in n passes. Using $x_n = 3y_{n-1}$, we obtain

$$x_{n-1} + x_n = 3^{n-1},$$

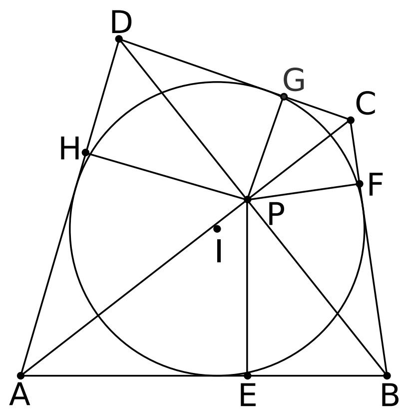
with $x_1 = 0$. Thus

$$\begin{aligned}
x_7 = 3^6 - x_6 = 3^6 - 3^5 + x_5 = 3^6 - 3^5 + 3^4 - x_4 = 3^6 - 3^5 + 3^4 - 3^3 + x_3 \\
= 3^6 - 3^5 + 3^4 - 3^3 + 3^2 - x_2 = 3^6 - 3^5 + 3^4 - 3^3 + 3^2 - 3 \\
= (2 \times 3^5) + (2 \times 3^3) + (2 \times 3) = 486 + 54 + 6 = 546.
\end{aligned}$$

5. Let $ABCD$ be a convex quadrilateral. Let the diagonals AC and BD intersect in P. Let PE, PF, PG and PH be the altitudes from P on to the sides AB, BC, CD and DA respectively. Show that $ABCD$ has an incircle if and only if

$$\frac{1}{PE} + \frac{1}{PG} = \frac{1}{PF} + \frac{1}{PH}.$$

Solution: Let $AP = p$, $BP = q$, $CP = r$, $DP = s$; $AB = a$, $BC = b$, $CD = c$ and $DA = d$. Let $\angle APB = \angle CPD = \theta$. Then $\angle BPC = \angle DPA = \pi - \theta$. Let us also write $PE = h_1$, $PF = h_2$, $PG = h_3$ and $PH = h_4$.



Observe that

$$h_1 a = pq \sin\theta, \quad h_2 b = qr \sin\theta, \quad h_3 c = rs \sin\theta, \quad h_4 d = sp \sin\theta.$$

Hence

$$\frac{1}{h_1} + \frac{1}{h_3} = \frac{1}{h_2} + \frac{1}{h_4}.$$

is equivalent to

$$\frac{a}{pq} + \frac{c}{rs} = \frac{b}{qr} + \frac{d}{sp}.$$

This is the same as

$$ars + cpq = bsp + dqr.$$

Thus we have to prove that $a+c = b+d$ if and only if $ars+cpq = bsp+dqr$. Now we can write $a + c = b + d$ as

$$a^2 + c^2 + 2ac = b^2 + d^2 + 2bd.$$

But we know that

$$a^2 = p^2 + q^2 - 2pq\cos\theta, \quad c^2 = r^2 + s^2 - 2rs\cos\theta$$
$$b^2 = q^2 + r^2 + 2qr\cos\theta, \quad d^2 = p^2 + s^2 + 2ps\cos\theta,$$

Hence $a + c = b + d$ is equivalent to

$$-pq\cos\theta + -rs\cos\theta + ac = ps\cos\theta + qr\cos\theta + bd.$$

Similarly, by squaring $ars + cpq = bsp + dqr$ we can show that it is equivalent to

$$-pq\cos\theta + -rs\cos\theta + ac = ps\cos\theta + qr\cos\theta + bd.$$

We conclude that $a + c = b + d$ is equivalent to $cpq + ars = bps + dqr$. Hence $ABCD$ has an in circle if and only if

$$\frac{1}{h_1} + \frac{1}{h_3} = \frac{1}{h_2} + \frac{1}{h_4}.$$

6. From a set of 11 square integers, show that one can choose 6 numbers $a^2, b^2, c^2, d^2, e^2, f^2$ such that

$$a^2 + b^2 + c^2 \equiv d^2 + e^2 + f^2 \pmod{12}.$$

Solution: The first observation is that we can find 5 pairs of squares such that the two numbers in a pair have the same parity. We can see this as follows:

Odd numbers	Even numbers	Odd pairs	Even pairs	Total pairs
0	11	0	5	5
1	10	0	5	5
2	9	1	4	5
3	8	1	4	5
4	7	2	3	5
5	6	2	3	5
6	5	3	2	5
7	4	3	2	5
8	3	4	1	5
9	2	4	1	5
10	1	5	0	5
11	0	5	0	5

Let us take such 5 pairs: say $(x_1^2, y_1^2), (x_2^2, y_2^2), \ldots, (x_5^2, y_5^2)$. Then $x_j^2 - y_j^2$ is divisible by 4 for $1 \le j \le 5$. Let r_j be the remainder when $x_j^2 - y_j^2$ is divisible by 3, $1 \le j \le 3$. We have 5 remainders r_1, r_2, r_3, r_4, r_5. But these can be 0, 1 or 2. Hence either one of the remainders occur 3 times or each of the remainders occur once. If, for example $r_1 = r_2 = r_3$, then 3 divides $r_1 + r_2 + r_3$; if $r_1 = 0, r_2 = 1$ and $r_3 = 2$, then again 3 divides $r_1 + r_2 + r_3$. Thus we can always find three remainders whose sum is divisible by 3. This means we can find 3 pairs, say, $(x_1^2, y_1^2), (x_2^2, y_2^2), (x_3^2, y_3^2)$ such that 3 divides $(x_1^2 - y_1^2) + (x_2^2 - y_2^2) + (x_3^2 - y_3^2)$. Since each difference is divisible by 4, we conclude that we can find 6 numbers $a^2, b^2, c^2, d^2, e^2, f^2$ such that

$$a^2 + b^2 + c^2 \equiv d^2 + e^2 + f^2 \pmod{12}.$$