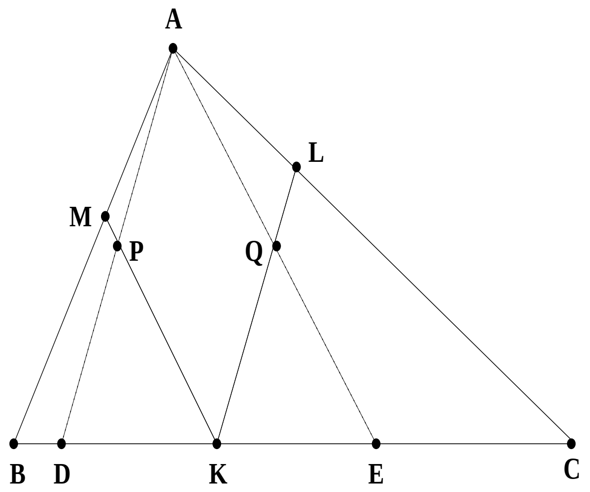
INMO-2000
Problems and Solutions

1. The in-circle of triangle ABC touches the sides BC, CA and AB in K, L and M respectively. The line through A and parallel to LK meets MK in P and the line through A and parallel to MK meets LK in Q. Show that the line PQ bisects the sides AB and AC of triangle ABC.

   Solution. : Let AP, AQ produced meet BC in D, E respectively.



   Since MK is parallel to AE, we have $\angle AEK = \angle MKB$. Since $BK = BM$, both being tangents to the circle from B, $\angle MKB = \angle BMK$. This with the fact that MK is parallel to AE gives us $\angle AEK = \angle MAE$. This shows that $MAEK$ is an isosceles trapezoid. We conclude that $MA = KE$. Similarly, we can prove that $AL = DK$. But $AM = AL$. We get that $DK = KE$. Since KP is parallel to AE, we get $DP = PA$ and similarly $EQ = QA$. This implies that PQ is parallel to DE and hence bisects AB, AC when produced.

   [The same argument holds even if one or both of P and Q lie outside triangle ABC.]

2. Solve for integers x, y, z:

$$x + y = 1 - z, \qquad x^3 + y^3 = 1 - z^2.$$

   Sol. : Eliminating z from the given set of equations, we get

$$x^3 + y^3 + \{1 - (x + y)\}^2 = 1.$$

This factors to

$$(x+y)(x^2-xy+y^2+x+y-2)=0.$$

Case 1. Suppose $x+y=0$. Then $z=1$ and $(x,y,z)=(m,-m,1)$, where m is an integer give one family of solutions.

Case 2. Suppose $x+y\neq 0$. Then we must have

$$x^2-xy+y^2+x+y-2=0.$$

This can be written in the form

$$(2x-y+1)^2+3(y+1)^2=12.$$

Here there are two possibilities:

$$2x-y+1=0, y+1=\pm 2; \qquad 2x-y+1=\pm 3, y+1=\pm 1.$$

Analysing all these cases we get

$$(x,y,z)=(0,1,0),(-2,-3,6),(1,0,0),(0,-2,3),(-2,0,3),(-3,-2,6).$$

3. If a,b,c,x are real numbers such that $abc\neq 0$ and

$$\frac{xb+(1-x)c}{a}=\frac{xc+(1-x)a}{b}=\frac{xa+(1-x)b}{c},$$

then prove that either $a+b+c=0$ or $a=b=c$.

Sol. : Suppose $a+b+c\neq 0$ and let the common value be λ. Then

$$\lambda=\frac{xb+(1-x)c+xc+(1-x)a+xa+(1-x)b}{a+b+c}=1.$$

We get two equations:

$$-a+xb+(1-x)c=0, \qquad (1-x)a-b+xc=0.$$

(The other equation is a linear combination of these two.) Using these two equations, we get the relations

$$\frac{a}{1-x+x^2}=\frac{b}{x^2-x+1}=\frac{c}{(1-x)^2+x}.$$

Since $1-x+x^2\neq 0$, we get $a=b=c$.

4. In a convex quadrilateral $PQRS$, $PQ = RS$, $(\sqrt{3}+1)QR = SP$ and $\angle RSP - \angle SPQ = 30^\circ$. Prove that

$$\angle PQR - \angle QRS = 90^\circ.$$

Sol. : Let [Fig] denote the area of Fig. We have

$$[PQRS] = [PQR] + [RSP] = [QRS] + [SPQ].$$

Let us write $PQ = p, QR = q, RS = r, SP = s$. The above relations reduce to

$$pq \sin \angle PQR + rs \sin \angle RSP = qr \sin \angle QRS + sp \sin \angle SPQ.$$

Using $p = r$ and $(\sqrt{3}+1)q = s$ and dividing by pq, we get

$$\sin \angle PQR + (\sqrt{3}+1) \sin \angle RSP = \sin \angle QRS + (\sqrt{3}+1) \sin \angle SPQ.$$

Therefore, $\sin \angle PQR - \sin \angle QRS = (\sqrt{3}+1)(\sin \angle SPQ - \sin \angle RSP)$.

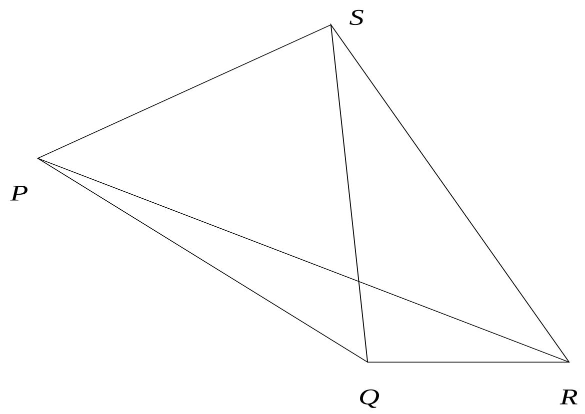


Fig. 2.

This can be written in the form

$$2 \sin \frac{\angle PQR - \angle QRS}{2} \cos \frac{\angle PQR + \angle QRS}{2}$$
$$= (\sqrt{3}+1) 2 \sin \frac{\angle SPQ - \angle RSP}{2} \cos \frac{\angle SPQ + \angle RSP}{2}.$$

Using the relations

$$\cos \frac{\angle PQR + \angle QRS}{2} = -\cos \frac{\angle SPQ + \angle RSP}{2}$$

and

$$\sin \frac{\angle SPQ - \angle RSP}{2} = -\sin 15^\circ = -\frac{(\sqrt{3}-1)}{2\sqrt{2}},$$

we obtain

$$\sin\frac{\angle PQR-\angle QRS}{2}=(\sqrt{3}+1)[-\frac{(\sqrt{3}-1)}{2\sqrt{2}}]=\frac{1}{\sqrt{2}}.$$

This shows that

$$\frac{\angle PQR-\angle QRS}{2}=\frac{\pi}{4}\quad\text{or}\quad\frac{3\pi}{4}.$$

Using the convexity of $PQRS$, we can rule out the latter alternative. We obtain

$$\angle PQR-\angle QRS=\frac{\pi}{2}.$$

5. Let a, b, c be three real numbers such that $1 \geq a \geq b \geq c \geq 0$. Prove that if λ is a root of the cubic equation $x^3 + ax^2 + bx + c = 0$ (real or complex),then $|\lambda| \leq 1$.

Sol. : Since λ is a root of the equation $x^3 + ax^2 + bx + c = 0$, we have

$$\lambda^3 = -a\lambda^2 - b\lambda - c.$$

This implies that

$$\begin{aligned}\lambda^4 &= -a\lambda^3 - b\lambda^2 - c\lambda \\ &= (1-a)\lambda^3 + (a-b)\lambda^2 + (b-c)\lambda + c\end{aligned}$$

where we have used again

$$-\lambda^3 - a\lambda^2 - b\lambda - c = 0.$$

Suppose $|\lambda| \geq 1$. Then we obtain

$$\begin{aligned}|\lambda|^4 &\leq (1-a)|\lambda|^3 + (a-b)|\lambda|^2 + (b-c)|\lambda| + c \\ &\leq (1-a)|\lambda|^3 + (a-b)|\lambda|^3 + (b-c)|\lambda|^3 + c|\lambda|^3 \\ &\leq |\lambda|^3.\end{aligned}$$

This shows that $|\lambda| \leq 1$. Hence the only possibility in this case is $|\lambda| = 1$. We conclude that $|\lambda| \leq 1$ is always true.

6. For any natural number n, $(n \geq 3)$, let $f(n)$ denote the number of non-congruent integer-sided triangles with perimeter n (e.g., $f(3) = 1, f(4) = 0, f(7) = 2$). Show that

$$\begin{aligned}&\text{(a)}\quad f(1999) > f(1996); \\ &\text{(b)}\quad f(2000) = f(1997).\end{aligned}$$

Sol. :

(a) Let a, b, c be the sides of a triangle with $a+b+c = 1996$, and each being a positive integer. Then $a+1, b+1, c+1$ are also sides of a triangle with perimeter 1999 because

$$a < b + c \quad\Longrightarrow\quad a + 1 < (b + 1) + (c + 1),$$

and so on. Moreover $(999, 999, 1)$ form the sides of a triangle with perimeter 1999, which is not obtainable in the form $(a+1, b+1, c+1)$ where a, b, c are the integers and the sides of a triangle with $a + b + c = 1996$. We conclude that $f(1999) > f(1996)$.

(b) As in the case (a) we conclude that $f(2000) \geq f(1997)$. On the other hand, if x, y, z are the integer sides of a triangle with $x+y+z = 2000$, and say $x \geq y \geq z \geq 1$, then we cannot have $z = 1$; for otherwise we would get $x + y = 1999$ forcing x, y to have opposite parity so that $x - y \geq 1 = z$ violating triangle inequality for x, y, z. Hence $x \geq y \geq z > 1$. This implies that $x - 1 \geq y - 1 \geq z - 1 > 0$. We already have $x < y + z$. If $x \geq y + z - 1$, then we see that $y + z - 1 \leq x < y + z$, showing that $y + z - 1 = x$. Hence we obtain $2000 = x + y + z = 2x + 1$ which is impossible. We conclude that $x < y + z - 1$. This shows that $x - 1 < (y - 1) + (z - 1)$ and hence $x - 1, y - 1, z - 1$ are the sides of a triangle with perimeter 1997. This gives $f(2000) \leq f(1997)$. Thus we obtain the desired result.