Indian National Astronomy Olympiad – 2016

Question Paper INAO – 2016

Roll Number: ☐☐☐☐ - ☐☐☐☐ - ☐☐☐☐ Date: 30th January 2016

Duration: **Three Hours** Maximum Marks: 100

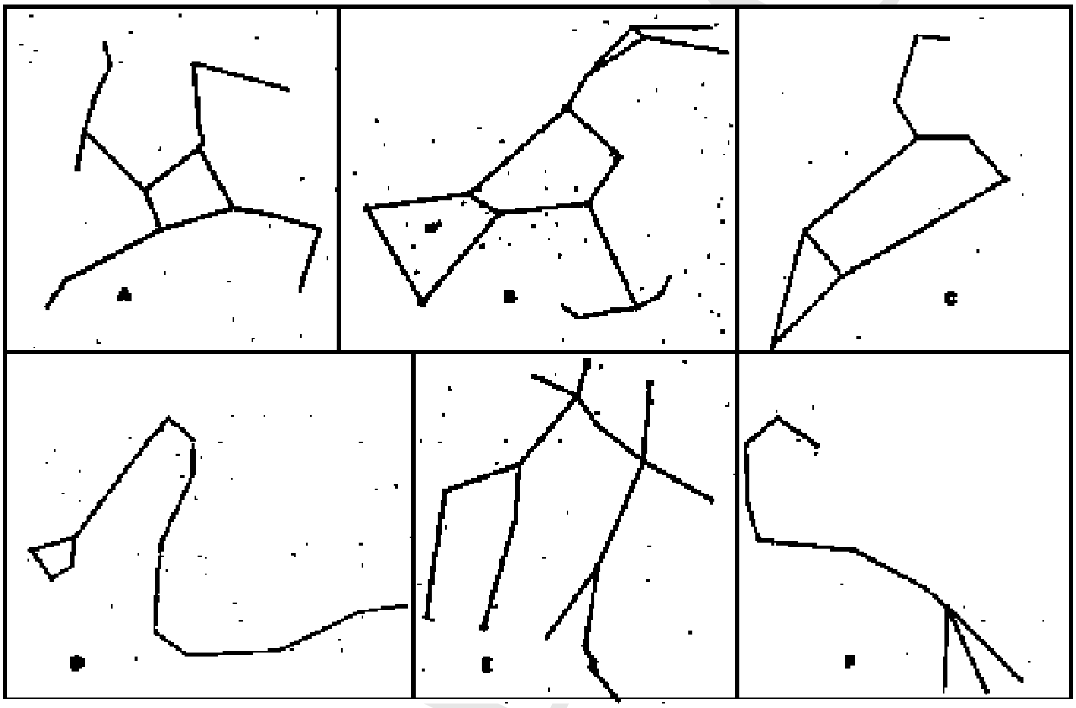
Please Note:

- Please write your roll number on top of this page in the space provided.
- Before starting, please ensure that you have received a copy of the question paper containing total 3 pages (5 sides).
- There are total 9 questions. Maximum marks are indicated in front of each question.
- For all questions, the process involved in arriving at the solution is more important than the answer itself. Valid assumptions / approximations are perfectly acceptable. Please write your method clearly, explicitly stating all reasoning.
- Blank spaces are provided in the question paper for the rough work. No rough work should be done on the answer-sheet.
- No computational aides like calculators, log tables, slide rule etc. are allowed.
- **The answer-sheet must be returned to the invigilator**. You can take this question booklet back with you.
- Please be advised that tentative dates for the next stage are as follows:
  - Orientation Cum Selection Camp (Senior): 3rd May to 19th May 2016.

Useful Physical Constants

Mass of the Earth	M_E	$\approx$	6.0×10^{24} kg
Radius of the Earth	R_E	$\approx$	6.4×10^{6} m
Speed of Light	c	$\approx$	3.0×10^{8} m/s
Astronomical Unit	1 A. U.	$\approx$	1.5×10^{11} m
Solar Luminosity	$L_\odot$	$\approx$	3.8×10^{26} W
Gravitational Constant	G	$\approx$	$\frac{20}{3} \times 10^{-11}$ m^3 kg^{-1} s^{-2}
Avogadro constant	N_a	$\approx$	6.0×10^{23} mol^{-1}
Charge of an electron	e	$\approx$	1.6×10^{-19} C
Permeability of free space	μ_0	$\approx$	$4\pi \times 10^{-7}$ N A^{-2}
Stephan's Constant	σ	$=$	$\frac{17}{3} \times 10^{-8}\ Wm^{-2}K^{-4}$
Radius of Neptune	R_N	$=$	24 622 km
Average Density of Neptune,	$\bar{\rho_N}$	$=$	1.638 g cm^{-3}
Specific heat capacity of water		$=$	4.2 J g^{-1}C^{-1}
Average calorific value of LPG gas		$\approx$	5×10^{4} kJ kg^{-1}
Approximate calorific value of methane		$\approx$	5×10^{4} kJ kg^{-1}

1. (9 marks) *Satej* is a twelve year old boy. He loves astronomy. Recently, he attended a sky observation workshop, organized by the Amateur Astronomers Club at a hill station. During the observation session, he was introduced to Hercules, Taurus, Scorpius, Orion, Ursa Major, Gemini, Draco, Leo, Serpens and Virgo constellations. He had sketched the stick figures of some of these constellations in his note book. Unfortunately, he forgot to write the names of the constellations and is finding it difficult to identify the constellations. Can you help him to identify the constellations for his sketches?



Solution:

A - Hercules	B - Orion	C - Leo
D - Draco	E - Gemini	F - Scorpius

2. (12 marks) In desperate times of depleting energy resources, *Sandesh* comes with an ingenious idea. He wishes to use methane on Neptune to satisfy Earth's ever increasing energy needs but needs to work out a few details. Help him to see of his idea is a good one by solving the following questions.

   (a) Estimate the total number of domestic LPG cylinders that can be filled up with methane from Neptune.

   (b) Estimate the net domestic consumption of LPG cylinders in a year in India. Justify your approach in arriving at this number. We then extend the answer obtained to represent the earth.

   (c) What is the approximate time-period, that methane brought from Neptune, can sustain the earth? Is *Sandesh's* idea good enough to satisfy Earth's energy needs?

Solution:

(a) We start by calculating the total mass of Neptune.

$$M = \bar{\rho_N}\frac{4}{3}\pi R_N^3 \approx 1 \times 10^{26}\,\mathrm{kg}$$

We estimate the fraction of methane to be around $\alpha = 0.01$

[(0.001 − 0.1) is acceptable]

Mass of methane on Neptune = 1×10^{24} kg

The mass of gas present in a cylinder is around 14 kg

[10 − 20 kg is acceptable]

Total number of cylinders that can be filled up $\approx 10^{23}$. **(4 marks)**

(b) We need to find the number of cylinders consumed by, say, a family of 5 in a year.

It takes around 5 minutes to heat 1000 ml of water to 100^o Celsius on a standard gas stove (take for example, preparing tea to arrive at this number!).

Using the specific heat of water, we arrive at energy output rate of LPG cylinder at around 6.0×10^4 J/minute.

Assuming that an average family uses a gas stove for around 3 hours daily, we estimate daily energy consumption to be around 1×10^7 J. **(3 marks)**

A domestic LPG cylinder has about 14 kg of gas. Also stoves in general are not 100% efficient. Let us assume a efficiency of around 60%. Using the given value of calorific value, the net energy content of a cylinder comes to be around 4.2×10^8 J.

Using these values, we obtain the yearly consumption to be around 3.7×10^9 J. This leads to consumption of around 10 cylinders per year for a family of 5. (Obviously this number can vary depending on the approximations made).

Therefore, *each person* uses around 2 cylinders per year. **(3 marks)**

Assuming total population of India to be around 1.1 billion, that gives total domestic consumption to be around $\approx 2.2 \times 10^9$ cylinders per year. Assuming that average consumption of an Indian household applies to the world population we approximately estimate the total domestic consumption for the world to be 1.32×10^{10}. **(1 marks)**

(c) Using the solution of part (a) and (b), the estimated time period comes to be around 4×10^{12} years. Life's good. **(1 marks)**

3. (a) (5 marks) *Anjali* is quite particular about noting down the daily earnings she makes in her side business, run by her and her mother. She noticed that over a period of 5 consecutive days, the money she earned daily, was basically the date multiplied by 10. She found the total money earned over these 5 days to be Rs 630. What were the dates corresponding to those 5 consecutive days?

Solution:
We see that if we take the dates to be an arithmetic series we get $50\,x+100 = 630$ which gives us $x = 10.6$ **(1 mark for equation)** which is not an integer and hence can not be a date. So the dates are at the end of a month and at the beginning of another. Given the total the dates that work are Feb 28, Feb 29, Mar 1, Mar 2 and Mar 3. **(5 marks for correct answer with explanation, 4 marks for correct answer without explanation.**

(b) (5 marks) What is the maximum value of n for which 100! is divisible by 2^n

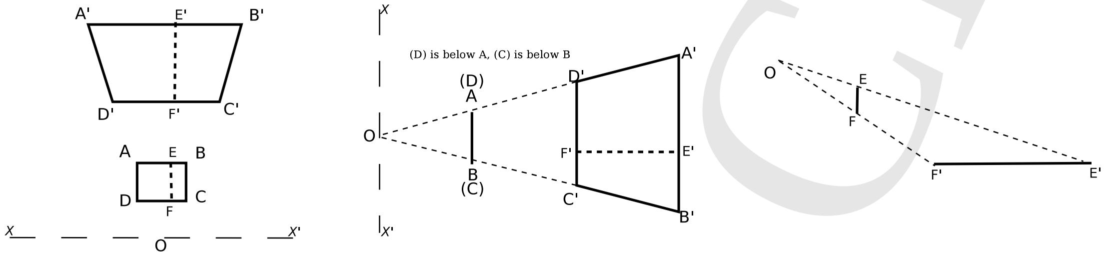
Solution:
We obtain the value of n in the following manner. For numbers up to 100, there are 50 even numbers, 25 multiples of 4, 12 multiples of 8, 6 multiples of 16, 3 multiples of 32 and 1 multiple of 64 which gives $n = 97$. **(5 marks)**

4. (10 marks) A boat is sailing on a river, parallel to a rectangular signboard propped up on the bank of the river. A search lamp on a flagpole on the top of the boat illuminates the signboard. The height of the lamp on the boat is such that complete shadow of the signboard is cast on the ground. As the boat moves, a sailor in the observation cabin of the boat starts seeing the clear shadow when the boat is at a

distance of about 100 m from the signboard. The boat passes the signboard at a closest distance of 60 m. What will be the ratio of the minimum area of the shadow to that of the maximum area of the shadow as the boat passes by? Explain your answer using very clear diagram/s.

Solution:

The shadow formed by a rectangle on the ground is always a trapezium, provided the rectangle is properly perpendicular to the ground.



Front View **Top View** **Side View**

In the figure, O is the point light source (search lamp) that moves along the line XX', ABCD is the rectangular signboard, A'B'C'D' is the shadow, with the vertices A', B', C' D' corresponding to the vertices A, B, C and D respectively. EF is the perpendicular distance between the sides AB and CD, while E'F' is the perpendicular distance between the sides A'B' and C'D' (in the shadow).

Front View: The shadow always remains a trapezium, although the shape of the trapezium changes as O moves. The area of the trapezium is the product of the sum of the parallel sides (A'B' + C'D') and the perpendicular distance between the two parallel sides (E'F'). Obviously, for the rectangle the area is product of the sides AB and BC (and EF = BC). **(5 marks)**

Top View: O moves along the line XX', which is parallel to AB. The vertex C is below B, and D is below A, hence these two vertices are not visible from the top view. By property of similar triangles, the ratio of A'B' to that of AB is always constant, and ditto for CD and C'D' (as long as XX' is parallel to AB). Hence, as O moves along the line XX', the position of A'B' changes but the length A'B' remains the same (as AB is a constant fixture), and ditto for C'D'. Thus, the sum of the parallel lines of the trapezium remains constant. **(3 marks)**

Side View: O is moving perpendicular to the plane of the paper. The angle $\angle EOF$ remains constant, which is equal to $\angle E'OF'$. Hence the ratio of EF to E'F' remains constant throughout, implying that E'F' remains constant throughout (as EF is a constant fixture). Thus the perpendicular distance between the parallel lines of the trapezium remains constant.

Therefore, the area of the shadow remains constant. Hence the ratio of the maximum area of the shadow to the minimum area of the shadow will be unity. **(2 marks)**

Note: Credit given for solutions with parallelograms provided the correct assumptions and approximations were given.

5. (6 marks) In the figure given below, we have 3 lenses with the same focal length 10 cm, with the distance of separation between each pair of the lenses being 30 cm. An object placed 20 cm from the first lens on the left produces an image 10 cm from the right of the rightmost lens (*See Figure*). After the lens assembly was moved by x cm to left, without moving the object, the image was again seen at the same location. Find x.

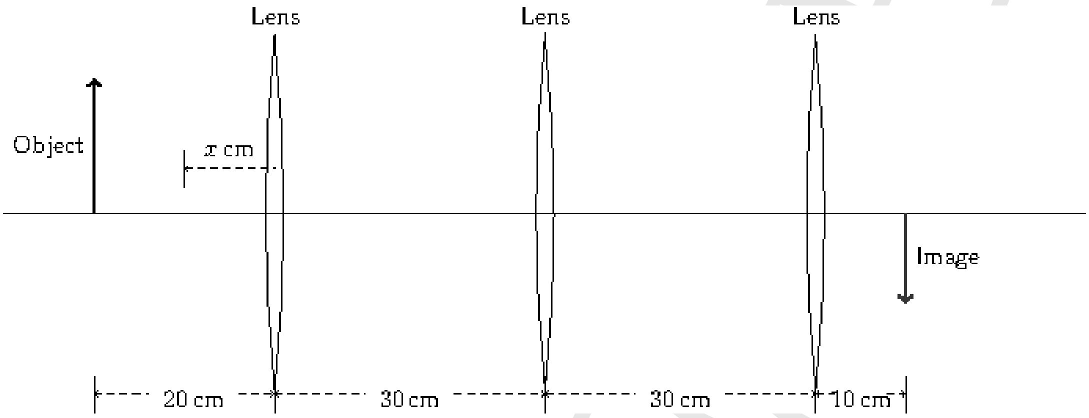


Figure 1

Solution:

Let u_1, v_1, and f_1 be the object, image and focal length of the first lens (from left). Similarly for the second and third lens. It is given that $f_1 = f_2 = f_3 = 10$cm. The lens equation $\frac{1}{u_i} + \frac{1}{v_i} = \frac{1}{f_i} = \frac{1}{10}$ has to be applied for each lens ($i = 1, 2, 3$). From the figure, for the first lens, $u_1 = 20 - x$. From the lens equation obtain v_1. Now, from the figure, $u_2 = 30 - v_1$, and then obtain v_2 from the lens equation. Repeat the procedure for the third lens. Hence, obtain the following values:-

$$u_1 = 20 - x, \quad v_1 = \frac{10(20-x)}{10-x},$$
$$u_2 = \frac{20(5-x)}{10-x}, \quad v_2 = \frac{-20(5-x)}{x},$$
$$u_3 = \frac{10(10+x)}{x}, \quad v_3 = 10 + x.$$

(4.5 marks)

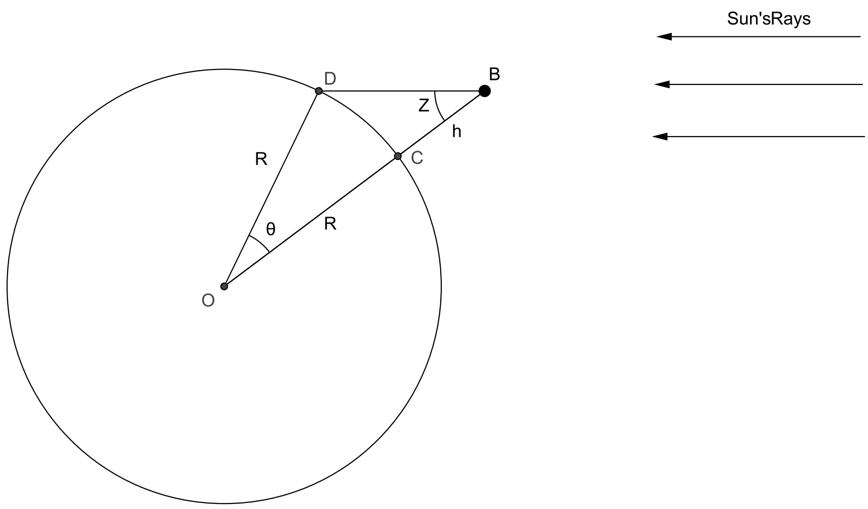
From the values obtained for u_1 and v_3, it can be seen that x can take any value from -10 to +20, i.e. after keeping the object fixed the lens set up can be moved towards the left by any distance up to 20 cm and towards right by any distance upto 10 cm, and the image will remain fixed at the same spot; the values of u_1 and v_3 change in such a manner that the sum $u_1 + v_3$ remains constant, and hence the sum of the total length of the set up inclduing the lens separation, i.e. $u_1 + v_3 + 30 + 30$ remains constant. Now, $u_1 + v_3 = 30$ always (for any value of $-10 < x < 20$, $x > 20$ is unphysical, and $x < -10$ will result in virtual image), hence the total distance between the object at u_1 and the image at v_3 remains

constant, the lens setup can be placed anywhere between the first object and final image. **(1.5 marks)**
Note: If done by conjugate foci argument only 2 to 3 marks.
Full credit to solution stating $0 < x < 20$ as the question explicitly states the lens assembly moves towards left. Also credit given to alternate logically correct explanation leading to the final answer.

6. (10 marks) *Ayush* measures the length of the shadow of a vertical **meter** stick in Kilimanjaro National Park to be 42 cm. What is the length of shadow subtended by Mount Kilimanjaro at that time?

   The following constants and relations might be useful:
   Height of Mt. Kilimanjaro = 5895 m
   Radius of Earth = 6400 km
   $\arctan 0.42 = 22.782°$
   $\sin 22.782° = 0.3872$
   $\arcsin 0.3876 = 22.805°$
   Please note that height of Mt. Kilimanjaro is not negligible as compared to radius of the earth. It is obviously assumed that base of Mount Kilimanjaro is small enough for the shadow to fall on the earth itself.

Solution:



If z represents zenith angle, R the radius of Earth and h denotes the height of Mt. Kilimanjaro, and θ the angle subtended by the shadow at Earth's center, we have by sine rule

$$\frac{h+R}{R} = \frac{\sin(z+\theta)}{\sin z}$$

which gives upon simplification

$$\theta = \arcsin\left(\left(\frac{h}{R}+1\right)\sin z\right) - z$$

Using values of the constants, you shall have

$$\theta \approx \arcsin((1+10^{-3})\ 0.3872) - 22.782^\circ$$

$$\theta \approx \arcsin(0.3872 + 0.0003872) - 22.782^\circ = 0.023^\circ$$

Thus the length, l, of shadow subtended is given by θR, where θ is measured in radian units.

$$l = \frac{0.023^\circ}{360^\circ}\ 2\ \frac{22}{7}\ 6400\,\text{km} \approx 2.53\,\text{km}.$$

(process: 6 marks, final answer: 4 marks)

Note : If the student uses similar triangles to estimate shadow to be $l = 0.42\,\text{m} \times 5895\,\text{m} = 2.48\,\text{km}$; 3 to 5 marks.

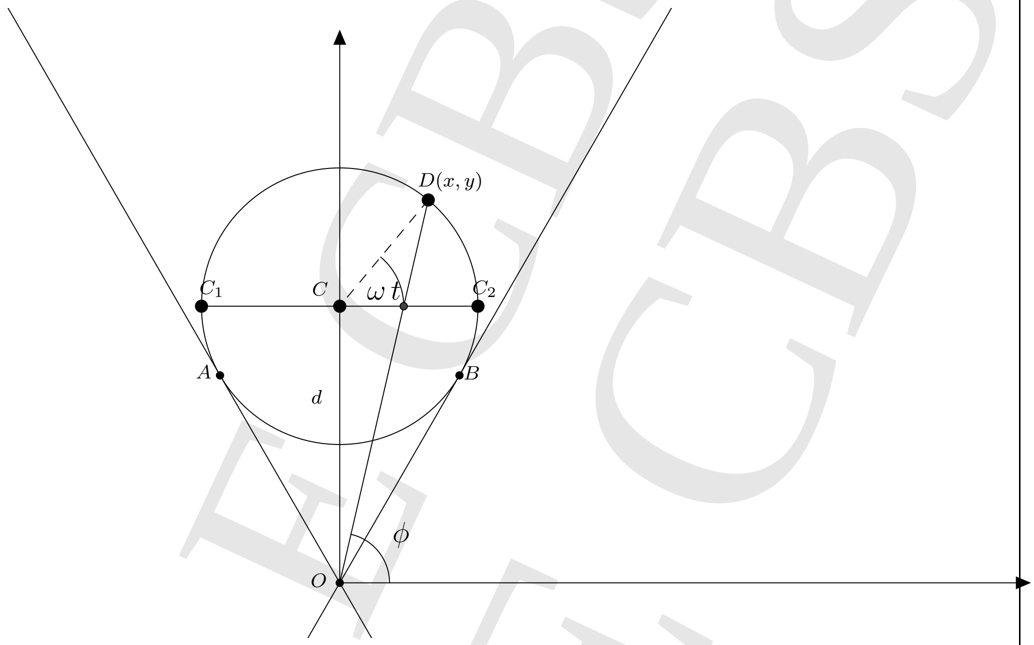
7. (16 marks) A whistle in its rest frame has a frequency of f_0 Hz. The whistle is kept on the outermost edge of a Merry-Go round (MGR) of radius a which rotates at a constant angular velocity ω. Find out the frequency heard f by a stationary listener standing on the ground at a distance d from the center of the MGR and also sketch qualitatively the graph of f v/s t when

   1. The listener is very far from the MGR.
   2. The listener is standing close to the MGR.
   3. The listener is standing close to the MGR and the MGR starts moving such that its center is revolving around the listener with angular speed Ω in the same (say clockwise) direction as the clockwise rotation of the MGR.
   4. The listener is standing close to the MGR and the MGR starts moving such that its center is revolving around the listener with angular speed Ω in the opposite (anti-clockwise) direction to the clockwise rotation of the MGR.

   Further calculate the time elapsed between the maximum frequency and minimum frequency hearings in each of the cases.

Solution:

The listener is standing at the origin of the coordinate system, $O(0,0)$, while the whistle is at $D(x,y)$ following the trajectory of a circle with centre C,in the rest frame of the listener.

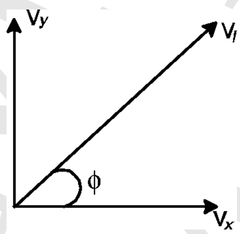


The maximum velocity of the whistle pointing away and towards the listener is not at points C_1 and C_2, but at points B and A respectively. Therefore,

$$\tan\phi = \frac{y}{x} = \frac{a\sin(\omega t) + d}{a\cos(\omega t)}$$

The velocity components of the whistle are

$$V_x = -a\omega\sin(\omega t)$$
$$V_y = a\omega\cos(\omega t)$$



Hence the net velocity in the direction OD (pointing away) of listener is

$$\begin{aligned} V_l &= V_y\sin\phi + V_x\cos\phi \\ &= a\omega\cos(\omega t)\sin\phi - a\omega\sin(\omega t)cos\phi \\ &= cos\phi[a\omega\cos(\omega t)\tan\phi - a\omega\sin(\omega t)] \\ &= \omega d\cos\phi \\ &= \frac{\omega a d\cos(\omega t)}{\sqrt{a^2 + d^2 + 2ad\sin(\omega t)}}. \end{aligned}$$

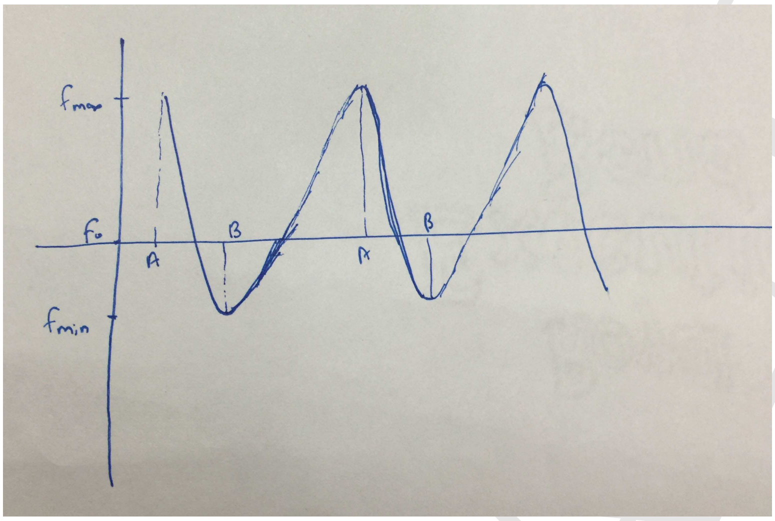
(7 marks)

This velocity is maximum $V_l^{max} = a\omega$ when $\sin\omega t = a/d$ at point B and when $\sin(\pi + \omega t) = a/d$ at point A. **(1 mark)**

The Doppler shifted frequency is then

$$f = \frac{c}{(c - V_l)} f_0$$

where c is the speed of sound. **(1 mark)**



The qualitative graph of f v/s t is as shown. **(2 marks)**

For case (1) , $d \gg a$, then points B and A will coincide with C_2 and C_1. The time elapsed will be simply π/ω.

For case (2), the time elapsed will be $\dfrac{\pi - 2\sin^{-1}(a/d)}{\omega}$. **(1 mark)**

Case (3) and (4) will be identical to case (2), as addition of any tangential component of velocity does not change the frequency. **(4 marks)**

8. (12 marks) *Dhruv* has been accidentally locked in a container that is 2.5 m long, 1.5 m wide and 6 m high. While waiting to be rescued, he throws a ball at the wall opposite to him and watches it hit the wall, bounce elastically back and forth between the walls before it hits the floor again. The vertical plane in which the ball travels is parallel to the other walls. If the ball is thrown from the level of the floor with a speed of $20\,\mathrm{ms^{-1}}$, the launch angle with respect to vertical being 60° and that he is leaning against one wall with the opposite wall being 1.5 m away, please calculate the following:-
   (a) What would be the maximum height reached?
   (b) After how long does the ball hit the flooor again?
   (c) How many times would the ball hit the walls before it reached the maximum height?

(d) With respect to the wall that he is leaning on, how far from wall would the ball be when it reaches the maximum height?

Solution:

(a) The collisions are elastic, hence the energy is conserved and the vertical component of the velocity remains unaffected due to the collisions. Let u_y be the initial vertical component of the velocity. Hence, $u_y = 20\cos(60^o) = 10\,\mathrm{ms}^{-1}$. Therefore, the maximum height, h, reached is (where g is gravitational acceleration on the surface of Earth and its direction is opposite to that of u_y)

$$h = \frac{u_y^2}{2g} = \frac{10^2}{2\times 10} = 5\,\mathrm{m} \qquad \textbf{(3 marks)}$$

(b) Since u_y is opposite in direction to that of g, the sign of g should be negative, considering the upward direction as positive. For the ball to come back to floor, the height (displacement) h is 0 (zero). Hence,

$$0 = u_y t - \frac{1}{2}gt^2$$
$$10t = \frac{1}{2}10t^2. \qquad \textbf{(3 marks)}$$

The trivial solution $t = 0$ corresponds to the intial condition. The other solution, $t = 2\,\mathrm{s}$ is the time after which the ball will hit the floor again.

(c) The time taken to reach the maximum height is 1 s. The horizontal distance travelled, d_x, in 1 s is $u_x \times 1$, where u_x is the horizontal component of the launch velocity, hence $u_x = 20\sin(60^o)$, therefore, $u_x = 20 \times (1.732/2) = 17.32\,\mathrm{ms}^{-1}$. Hence, the horizontal distance travelled is $d_x = 17.32 \times 1 = 17.32\,\mathrm{m}$. Now, the total number of collisions, n, with the walls which are at a distance of 1.5 m is given by $n = \mathrm{quotient}[17.32,1.5]$, giving $n = 11$. **(3 marks)**

(d) Before reaching the maximum height, the number of collisions with the wall is $n = 11$. The final collision was from the opposite wall. The distance travelled until the last (11^{th}) collision is $11 \times 1.5 = 16.5\,\mathrm{m}$. Therefore, distance from the opposite wall (at maximum height), $s_{opp} = 17.32 - 16.50 = 0.82\,\mathrm{m}$. The distance from the intial wall will be $s = 1.5 - 0.82 = 0.68\,\mathrm{m}$. **(3 marks)**

9. (15 marks) Ameya wishes to develop a simple device, which, when connected to the prime focus of a telescope, will give a direct reading of the apparent magnitude of the object that the telescope is pointing at. He has with him a set of data points relating the current obtained when light of a particular intensity falls on the device as given in table below. He is confident that once he gets the calibration curve, relating Intensity to current, he can make another calibration curve, one that connects the apparent magnitude of the object that he is looking at through the telescope, to the current that he is measuring in his device. Please help him in drawing both the curves. Also, estimate the current reading for Venus and Sirius if their apparent magnitudes are -4.43 and -1.46 respectively.

   Note: The apparent magnitude m of a celestial object is a measure of its brightness as seen by an observer on Earth, adjusted to the value it would have in the absence of the atmosphere. The brighter an object appears, the lower its magnitude value (i.e. inverse relation). Also, the magnitude scale is logarithmic: a difference of one in magnitude corresponds to a change in brightness by a factor of about 2.5. The mathematical relation is given by $m_1 - m_2 = 2.5 \log\left(\frac{I_2}{I_1}\right)$. The reference point for the apparent magnitude scale is the star Vega. The intensity of Vega incident on earth is $2.18 \times 10^{-8}\,\mathrm{W\,m^{-2}}$.

   Some useful data for this question.

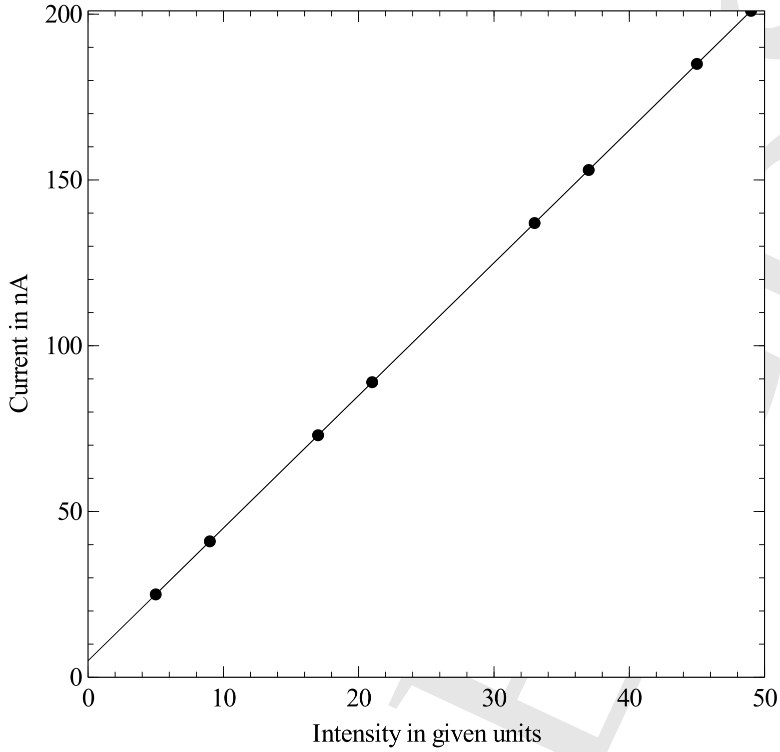
Intensity	Current
$10^{-8}\,\mathrm{W\,m^{-2}}$	nA
5	25
9	41
17	73
21	89
33	137
37	153
45	185
49	201

x	$\log(x)$
2	0.301
3	0.477
4	0.602
5	0.699
6	0.778
7	0.845
8	0.903
9	0.954

Antilog values
$10^{4.43} = 26195.34$
$10^{1.46} = 28.84$
$10^{1.772} = 59.16$
$10^{.584} = 3.84$
$10^{0.443} = 2.77$
$10^{0.146} = 1.40$

Solution:

We first plot the Current vs Intensity graph from the data given. We get a straight line graph whose equation is given by $y = 4\,x + 5$ where y is current in nA and x is intensity in $10^{-8}\mathrm{W\,m^{-2}}$. The y-intercept which represents the dark current is $c = 5$nA.



(3 marks for complete Graph with Data Points, labels and scale)

We get a straight line graph whose equation is given by $y = 4\,x + 5$ where y is current in nA and x is intensity in $10^{-8}\mathrm{W\,m^{-2}}$. The y-intercept which represents the dark current is $c = 5\mathrm{nA}$

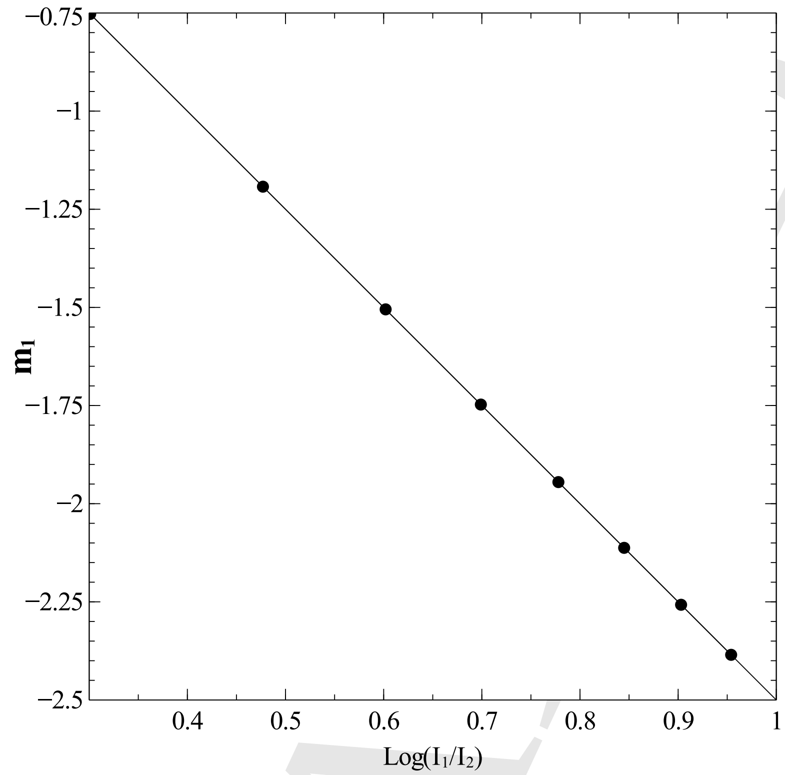
(2 marks for equation with explanation on how it was obtained)

The Intensity of Vega which is the reference point is $2.1810^{-8}\mathrm{W\,m^{-2}}$. This gives a current reading of 13.72nA

We then plot the graph for $m_1 - m_2 = -2.5\log\left(\frac{I_1}{I_2}\right)$. This is done as follows. We plot $\log\left(\frac{I_1}{I_2}\right)$ on the x-axis and m_1 on the y-axis. Since we are given that Vega is the reference point, we take all intensities as multiples of $I_{vega} = I_2$ and so we can put $m_2 = 0$. The equation for the graph is $u = -2.5\,v$. The data points needed for this plot can be calculated by taking all intensities as multiples of the reference intensity. The data points obtained are listed below.

v	u
0.301	-0.7525
0.477	-1.1925
0.602	-1.505
0.699	-1.7475
0.778	-1.945
0.845	-2.1125
0.903	-2.2575
0.954	-2.385

The plot for these points is given below.



(7 marks for complete straight line graph with data points, labels and scale)

Note: Extrapolation (and interpolation) from a non-linear curve does not get credit.

We now proceed to estimate the current reading for Venus and Sirius as follows. For Sirius the magnitude is given by $m_1 = -1.46$. So from the graph this corresponds to an x-axis value of $x = 0.584$. From the data given we obtain $\frac{I_1}{I_2} = 3.84$.

Since y_1 is current due to light intensity from vega we have $y_2 - c = 3.84 \times (13.72 - 5) = 33.49$nA. Therefore current reading for Sirius would be $y_2 = 33.49 + 5 = 38.49$nA

(2 marks for answer including explanation)

For Venus the magnitude is given by $m_1 = -4.43$. So from the graph this corresponds to an x-axis value of $x = 1.772$. From the data given we obtain $\frac{I_1}{I_2} = 59.16 \approx 60$.

Since y_1 is current due to light intensity from vega we have $y_2 - c = 59.16 \times (13.72 - 5) = 515.88$nA. Therefore current reading for Venus would be $y_2 = 515.88 + 5 = 520.88$nA

(3 marks for answer including explanation)

Notes for Junior Group

- Question 7: The formula for Doppler shift of frequency (for a stationary listener) is as follows:-
  $f = \frac{c}{c+v_s} f_0$
  where c = velocity of sound, v_s = parallel component of velocity of source
  f_0 = emitted frequency of source, f = observed frequency
- Question 9: The apparent magnitude of Vega, $m = 0$.