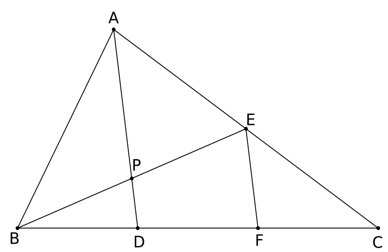
Problems and Solutions: CRMO-2012, Paper 1

1. Let ABC be a triangle and D be a point on the segment BC such that $DC = 2BD$. Let E be the mid-point of AC. Let AD and BE intersect in P. Determine the ratios BP/PE and AP/PD.



Solution: Let F be the midpoint of DC, so that D, F are points of trisection of BC. Now in triangle CAD, F is the mid-point of CD and E is that of CA. Hence $CF/FD = 1 = CE/EA$. Thus $EF \parallel AD$. Hence we find that $EF \parallel PD$. Hence $BP/PE = BD/DF$. But $BD = DF$. We obtain $BP/PE = 1$.

In triangle ACD, since $EF \parallel AD$ we get $EF/AD = CF/CD = 1/2$. Thus $AD = 2EF$. But $PD/EF = BD/BF = 1/2$. Hence $EF = 2PD$. Therefore
This gives
$$AP = AD - PD = 3PD.$$
We obtain $AP/PD = 3$.
(Coordinate geometry proof is also possible.)

2. Let a, b, c be positive integers such that a divides b^3, b divides c^3 and c divides a^3. Prove that abc divides $(a+b+c)^{13}$.

Solution: If a prime p divides a, then $p \mid b^3$ and hence $p \mid b$. This implies that $p \mid c^3$ and hence $p \mid c$. Thus every prime dividing a also divides b and c. By symmetry, this is true for b and c as well. We conclude that a, b, c have the same set of prime divisors.

Let $p^x \parallel a$, $p^y \parallel b$ and $p^z \parallel c$. (Here we write $p^x \parallel a$ to mean $p^x \mid a$ and $p^{x+1} \not| a$.) We may assume $\min\{x, y, z\} = x$. Now $b \mid c^3$ implies that $y \leq 3z$; $c \mid a^3$ implies that $z \leq 3x$. We obtain
$$y \leq 3z \leq 9x.$$
Thus $x + y + z \leq x + 3x + 9x = 13x$. Hence the maximum power of p that divides abc is $x + y + z \leq 13x$. Since x is the minimum among x, y, z, whence p^x divides each of a, b, c. Hence p^x divides $a + b + c$. This implies that p^{13x} divides $(a+b+c)^{13}$. Since $x + y + z \leq 13x$, it follows that p^{x+y+z} divides $(a+b+c)^{13}$. This is true of any prime p dividing a, b, c. Hence abc divides $(a+b+c)^{13}$.

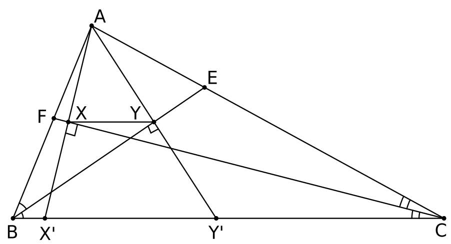
3. Let a and b be positive real numbers such that $a + b = 1$. Prove that
$$a^a b^b + a^b b^a \leq 1.$$

Solution: Observe
$$1 = a + b = a^{a+b} b^{a+b} = a^a b^b + b^a b^b.$$
Hence
$$1 - a^a b^b - a^b b^a = a^a b^b + b^a b^b - a^a b^b - a^b b^a = (a^a - b^a)(a^b - b^b)$$
Now if $a \leq b$, then $a^a \leq b^a$ and $a^b \leq b^b$. If $a \geq b$, then $a^a \geq b^a$ and $a^b \geq b^b$. Hence the product is nonnegative for all positive a and b. It follows that
$$a^a b^b + a^b b^a \leq 1.$$

4. Let $X = \{1, 2, 3, \ldots, 10\}$. Find the the number of pairs $\{A, B\}$ such that $A \subseteq X$, $B \subseteq X$, $A \neq B$ and $A \cap B = \{2, 3, 5, 7\}$.

Solution: Let $A \cup B = Y$, $B \setminus A = M$, $A \setminus B = N$ and $X \setminus Y = L$. Then X is the disjoint union of M, N, L and $A \cap B$. Now $A \cap B = \{2, 3, 5, 7\}$ is fixed. The remaining six elements $1, 4, 6, 8, 9, 10$ can be distributed in any of the remaining sets M, N, L. This can be done in 3^6 ways. Of these if all the elements are in the set L, then $A = B = \{2, 3, 5, 7\}$ and which this case has to be deleted. Hence the total number of pairs $\{A, B\}$ such that $A \subseteq X$, $B \subseteq X$, $A \neq B$ and $A \cap B = \{2, 3, 5, 7\}$ is $3^6 - 1$.

5. Let ABC be a triangle. Let BE and CF be internal angle bisectors of $\angle B$ and $\angle C$ respectively with E on AC and F on AB. Suppose X is a point on the segment CF such that $AX \perp CF$; and Y is a point on the segment BE such that $AY \perp BE$. Prove that $XY = (b + c - a)/2$ where $BC = a$, $CA = b$ and $AB = c$.



Solution: Produce AX and AY to meet BC is X' and Y' respectively. Since BY bisects $\angle ABY'$ and $BY \perp AY'$ it follows that $BA = BY'$ and $AY = YY'$. Similarly, $CA = CX'$ and $AX = XX'$. Thus X and Y are mid-points of AX' and AY' respectively. By mid-point theorem $XY = X'Y'/2$. But

$$X'Y' = X'C + Y'B - BC = AC + AB - BC = b + c - a.$$

Hence $XY = (b + c - a)/2$.

6. Let a and b be real numbers such that $a \neq 0$. Prove that not all the roots of $ax^4 + bx^3 + x^2 + x + 1 = 0$ can be real.

Solution: Let α_1, α_2, α_3, α_4 be the roots of $ax^4 + bx^3 + x^2 + x + 1 = 0$. Observe none of these is zero since their product is $1/a$. Then the roots of $x^4 + x^3 + x^2 + bx + a = 0$ are

$$\beta_1 = \frac{1}{\alpha_1}, \beta_2 = \frac{1}{\alpha_2}, \beta_3 = \frac{1}{\alpha_3}, \beta_4 = \frac{1}{\alpha_4}.$$

We have

$$\sum_{j=1}^{4} \beta_j = -1, \quad \sum 1 \leq j < k \leq 4 \beta_j \beta_k = 1.$$

Hence

$$\sum_{j=1}^{4} \beta_j^2 = \left(\sum_{j=1}^{4} \beta_j \right)^2 - 2 \left(\sum_{1 \leq j < k \leq 4} \beta_j \beta_k \right) = 1 - 2 = -1.$$

This shows that not all β_j can be real. Hence not all α_j's can be real.

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