1. Find the number of eight-digit numbers the sum of whose digits is 4.

   Solution. We need to find the number of 8-tuples $(a_1, a_2, \ldots, a_8)$ of non-negative integers such that $a_1 \geq 1$ and $a_1 + a_2 + \cdots + a_8 = 4$. If $a_1 = 1$, then there are three possibilities: either exactly three among $a_2, a_3, \ldots, a_7$ equal 1 and the rest equal zero, or five of them are zero and the other two equal 1 and 2, or six of them are zero and the other equals 3. In the first case, there are $\binom{7}{3} = 35$ such 8-tuples, in the second case there are $\binom{7}{2} \times 2 = 42$ such 8-tuples and in the third case there are 7 such 8-tuples. If $a_1 = 2$ then either six of $a_2, a_3, \ldots, a_7$ are zero and the other equals two, or five of them are zero and the remaining two both equal 1. In the former case, there are 7 such 8-tuples and in the latter case there are $\binom{7}{2} = 21$ such 8-tuples. If $a_1 = 3$ then exactly six of $a_2, a_3, \ldots, a_7$ are zero and the other equals one. There are 7 such 8-tuples. Finally, there is one 8-tuple in which $a_1 = 4$. Thus, in total, there are 120 such 8-tuples. $\square$

2. Find all 4-tuples (a, b, c, d) of natural numbers with $a \leq b \leq c$ and $a! + b! + c! = 3^d$.

   Solution. Note that if $a > 1$ then the left-hand side is even, and therefore $a = 1$. If $b > 2$ then 3 divides $b! + c!$ and hence 3 does not divide the left-hand side. Therefore $b = 1$ or $b = 2$.

   If $b = 1$ then $c! + 2 = 3^d$, so $c < 2$ and hence $d = 1$. If $b = 2$ then $c! = 3^d - 3$. Note that $d = 1$ does not give any solution. If $d > 1$ then 9 does not divide $c!$, so $c < 6$. By checking the values for $c = 2, 3, 4, 5$ we see that $c = 3$ and $c = 4$ are the only two solutions. Thus $(a, b, c, d) = (1, 1, 1, 1), (1, 2, 3, 2)$ or $(1, 2, 4, 3)$. $\square$

3. In an acute-angled triangle ABC with $AB < AC$, the circle Γ touches AB at B and passes through C intersecting AC again at D. Prove that the orthocentre of triangle ABD lies on Γ if and only if it lies on the perpendicular bisector of BC.

   Solution. Note that $\angle ADB = \angle B$ and hence triangles ADB and ABC are similar. In particular, ABD is an acute-angled triangle. Let H denote the orthocenter of triangle ABD. Then $\angle BHD = 180° - \angle A$.

   Suppose that H lies on Γ. Since $AB < AC$ the point D lies on the segment AC and $\angle C = 180° - \angle BHD = \angle A$. Therefore BH is the perpendicular bisector of AC. Hence $\angle HBC = \angle ABC = \angle HCB$, so H lies on the perpendicular bisector of BC.

   Conversely, suppose that H lies on the perpendicular bisector of BC. Then $\angle HCB = \angle HBC = 90° - \angle C$. Since $\angle ABD = \angle C$ it follows that $\angle HDB = 90° - \angle C$. Since $\angle HCB = \angle HDB$ we have that H lies on Γ. $\square$

4. A polynomial is called a *Fermat polynomial* if it can be written as the sum of the squares of two polynomials with integer coefficients. Suppose that $f(x)$ is a Fermat polynomial such that $f(0) = 1000$. Prove that $f(x) + 2x$ is not a Fermat polynomial.

   Solution. Let $p(x)$ be a Fermat polynomial such that $p(0)$ is divisible by 4. Suppose that $p(x) = g(x)^2 + h(x)^2$ where $g(x)$ and $h(x)$ are polynomials with integer coefficients. Therefore $g(0)^2 + h(0)^2$ is divisble by 4. Since $g(0)$ and $h(0)$ are integers, their squares are either 1 (mod 4) or 0 (mod 4). It therefore follows that $g(0)$ and $h(0)$ are even. Therefore the coefficents of x in $g(x)^2$ and in $h(x)^2$ are both divisible by 4. In particular, the coefficient of x in a Fermat polynomial $p(x)$, with $p(0)$ divisible by 4, is divisible by 4. Thus if $f(x)$ is a Fermat polynomial with $f(0) = 1000$ then $f(x) + 2x$ cannot be a Fermat polynomial.

□

5. Let ABC be a triangle which it not right-angled. Define a sequence of triangles $A_iB_iC_i$, with $i \geq 0$, as follows: $A_0B_0C_0$ is the triangle ABC; and, for $i \geq 0$, $A_{i+1}, B_{i+1}, C_{i+1}$ are the reflections of the orthocentre of triangle $A_iB_iC_i$ in the sides B_iC_i, C_iA_i, A_iB_i, respectively. Assume that $\angle A_m = \angle A_n$ for some distinct natural numbers m, n. Prove that $\angle A = 60°$.

Solution. The statement of the problem as stated is not correct. We give below the reason, and we shall also give the condition under which the statement becomes true.

Let P, Q, R denote the reflections of H with respect BC, CA, AB, respectively. Then P, Q, R lie on the circumcircle of the triangle. If ABC is an acute-angled triangle then $\angle QPR = \angle QPA + \angle RPA = \angle QCA + \angle RBA = 180° - 2\angle A$. Similarly, if $\angle A$ is obtuse then we get $\angle QPR = 2\angle A - 180°$. Therefore, for example, if $\angle A = 180°/7$ and $\angle B\angle C = 540°/7$ then we get that $\angle A_3 = 180°/7 = \angle A_0$. Therefore the statement of the problem is not correct.

However, the statement is correct provided all the triangles $A_iB_iC_i$ are acute-angled. Under this assumption we give below a proof of the statement.

Let α, β, γ denote the angles of T_0. Let $f_k(x) = (-2)^k x - ((-2)^k - 1)60°$. We claim that the angles of T_k are $f_k(\alpha), f_k(\beta)$ and $f_k(\gamma)$. Note that this claim is true for $k = 0$ and $k = 1$. It is easy to check that $f_{k+1}(x) = 180° - 2f_k(x)$, so the claim follows by induction.

If $T_m = T_n$, then $f_m(\alpha) = f_n(\alpha)$, so $\alpha((-2)^m - (-2)^n) = 60°((-2)^m - (-2)^n)$. Therefore, since $m \neq n$, it follows that $\alpha = 60°$. □

6. Let $n \geq 4$ be a natural number. Let $A_1A_2 \cdots A_n$ be a regular polygon and $X = \{1, 2, \ldots, n\}$. A subset $\{i_1, i_2, \ldots, i_k\}$ of X, with $k \geq 3$ and $i_1 < i_2 < \cdots < i_k$, is called a *good subset* if the angles of the polygon $A_{i_1}A_{i_2} \cdots A_{i_k}$, when arranged in the increasing order, are in an arithmetic progression. If n is a prime, show that a **proper** good subset of X contains exactly four elements.

Solution. We note that every angle of $A_{i_1}A_{i_2} \cdots A_{i_k}$ is a multiple of π/n. Suppose that these angles are in an arithmetic progression. Let r and s be non-negative integers such that $\pi r/n$ is the smallest angle in this progression and $\pi s/n$ is the common difference. Then we have

$$\frac{\pi}{n}(rk + sk(k-1)/2) = (k-2)\pi\,.$$

Therefore $rk + sk(k-1)/2 = (k-2)n$. Suppose that k is odd. Then k divides the left-hand side and k is coprime to $k-2$. Therefore k divides n. On the other hand if k is even then $k/2$ is coprime to $(k-2)/2$ and hence k divides $4n$. If n is prime and $k < n$ then it follows that k divides 4. Since $k > 2$, we have proved that $k = 4$. □

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