1. Let ABC be an acute angled triangle. The circle Γ with BC as diameter intersects AB and AC again at P and Q, respectively. Determine $\angle BAC$ given that the orthocenter of triangle APQ lies on Γ.

   Solution. Let K denote the orthocenter of triangle APQ. Since triangles ABC and AQP are similar it follows that K lies in the interior of triangle APQ.

   Note that $\angle KPA = \angle KQA = 90° - \angle A$. Since $BPKQ$ is a cyclic quadrilateral it follows that $\angle BQK = 180° - \angle BPK = 90° - \angle A$, while on the other hand $\angle BQK = \angle BQA - \angle KQA = \angle A$ since BQ is perpendicular to AC. This shows that $90° - \angle A = \angle A$, so $\angle A = 45°$. □

2. Let $f(x) = x^3 + ax^2 + bx + c$ and $g(x) = x^3 + bx^2 + cx + a$, where a, b, c are integers with $c \neq 0$. Suppose that the following conditions hold:

   (a) $f(1) = 0$;

   (b) the roots of $g(x)$ are squares of the roots of $f(x)$.

   Find the value of $a^{2013} + b^{2013} + c^{2013}$.

   Solution. Note that $g(1) = f(1) = 0$, so 1 is a root of both $f(x)$ and $g(x)$. Let p and q be the other two roots of $f(x)$, so p^2 and q^2 are the other two roots of $g(x)$. We then get $pq = -c$ and $p^2q^2 = -a$, so $a = -c^2$. Also, $(-a)^2 = (p+q+1)^2 = p^2+q^2+1+2(pq+p+q) = -b+2b = b$. Therefore $b = c^4$. Since $f(1) = 0$ we therefore get $1 + c - c^2 + c^4 = 0$. Factorising, we get $(c + 1)(c^3 - c^2 + 1) = 0$. Note that $c^3 - c^2 + 1 = 0$ has no integer root and hence $c = -1, b = 1, a = -1$. Therefore $a^{2013} + b^{2013} + c^{2013} = -1$. □

3. Find all primes p and q such that p divides $q^2 - 4$ and q divides $p^2 - 1$.

   Solution. Suppose that $p \leq q$. Since q divides $(p - 1)(p + 1)$ and $q > p - 1$ it follows that q divides $p + 1$ and hence $q = p + 1$. Therefore $p = 2$ and $q = 3$.

   On the other hand, if $p > q$ then p divides $(q - 2)(q + 2)$ implies that p divides $q + 2$ or $q - 2 = 0$. This gives either $p = q + 2$ or $q = 2$. In the former case it follows that that q divides $(q+2)^2 - 1$, so q divides 3. This gives the solutions $p > 2, q = 2$ and $(p, q) = (5, 3)$. □

4. Find the number of 10-tuples $(a_1, a_2, \ldots, a_{10})$ of integers such that $|a_1| \leq 1$ and

$$a_1^2 + a_2^2 + a_3^2 + \cdots + a_{10}^2 - a_1a_2 - a_2a_3 - a_3a_4 - \cdots - a_9a_{10} - a_{10}a_1 = 2\,.$$

   Solution. Let $a_{11} = a_1$. Multiplying the given equation by 2 we get

$$(a_1 - a_2)^2 + (a_2 - a_3)^2 + \cdots (a_{10} - a_1)^2 = 4\,.$$

   Note that if $a_i - a_{i+1} = \pm 2$ for some $i = 1, \ldots, 10$, then $a_j - a_{j+1} = 0$ for all $j \neq i$ which contradicts the equality $\sum_{i=1}^{10}(a_i - a_{i+1}) = 0$. Therefore $a_i - a_{i+1} = 1$ for exactly two values of i in $\{1, 2, \ldots, 10\}$, $a_i - a_{i+1} = -1$ for two other values of i and $a_i - a_{i+1} = 0$ for all other values of i. There are $\binom{10}{2} \times \binom{8}{2} = 45 \times 28$ possible ways of choosing these values. Note that $a_1 = -1, 0$ or 1, so in total there are $3 \times 45 \times 28$ possible integer solutions to the given equation. □

5. Let ABC be a triangle with $\angle A = 90^\circ$ and $AB = AC$. Let D and E be points on the segment BC such that $BD : DE : EC = 3 : 5 : 4$. Prove that $\angle DAE = 45^\circ$.

   Solution. Rotating the configuraiton about A by 90°, the point B goes to the point C. Let P denote the image of the point D under this rotation. Then $CP = BD$ and $\angle ACP = \angle ABC = 45^\circ$, so ECP is a right-angled triangle with $CE : CP = 4 : 3$. Hence $PE = ED$. It follows that $ADEP$ is a kite with $AP = AD$ and $PE = ED$. Therefore AE is the angular bisector of $\angle PAD$. This implies that $\angle DAE = \angle PAD/2 = 45^\circ$. □

6. Suppose that m and n are integers such that both the quadratic equations $x^2 + mx - n = 0$ and $x^2 - mx + n = 0$ have integer roots. Prove that n is divisible by 6.

   Solution. Let a be an integer. If a is not divisible by 3 then $a^2 \equiv 1 \pmod 3$, i.e., 3 divides $a^2 - 1$, and if a is odd then $a^2 \equiv 1 \pmod 8$, i.e., 8 divides $a^2 - 1$.

   Note that the discriminants of the two quadratic polynomials are both squares of integers. Let a and b be integers such that $m^2 - 4n = a^2$ and $m^2 + 4n = b^2$. Therefore $8n = b^2 - a^2$ and $2m^2 = a^2 + b^2$. If 3 divides m then 3 divides both a and b, so 3 divides n. On the other hand if 3 does not divide m then 3 does not divide a or b. Therefore 3 divides $b^2 - a^2$ and hence 3 divides n.

   If m is odd, then so is a, and therefore $4n = m^2 - a^2$ is divisible by 8, so n is even. On the other hand, if m is even then both a and b are even. Further $(m/2)^2 - n = (a/2)^2$ and $(m/2)^2 + n = (b/2)^2$, so $(b - a)/2$ is even. In particular, $n = (b^2 - a^2)/4$ is even. □

——— ⋆ ———