RMO 2025

Official Solutions

Problem 1.

(a) Let $n \geq 3$ be an integer. Find a configuration of n lines in the plane which has exactly

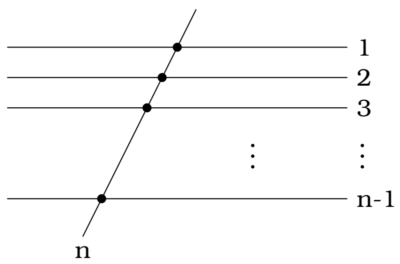
(i) $n-1$ distinct points of intersection;

(ii) n distinct points of intersection;

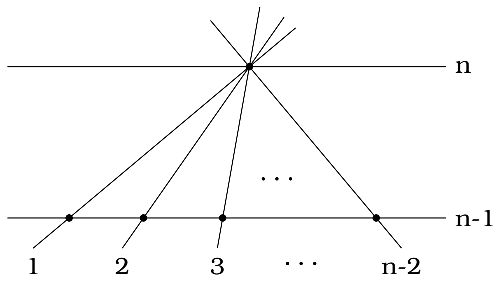
(b) Give configurations of n lines that have exactly $n+1$ distinct points of intersection for (i) $n=8$ and (ii) $n=9$.

Solution.

1. n **lines and** $n-1$ **distinct points of intersection**



Construction 1



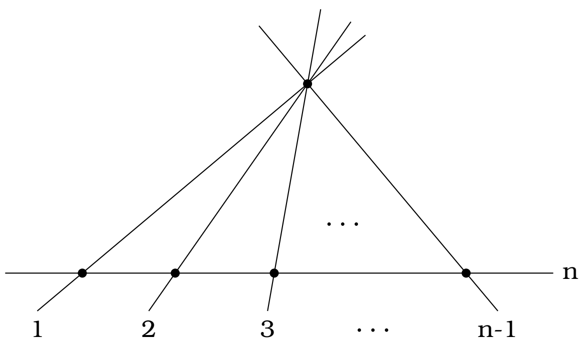
Construction 2

Either of these constructions provide us with an example of n lines with $n-1$ distinct points of intersection.

In **construction 1**, we have $n-1$ parallel lines, and one transverse line which intersects them all at exactly $n-1$ points.

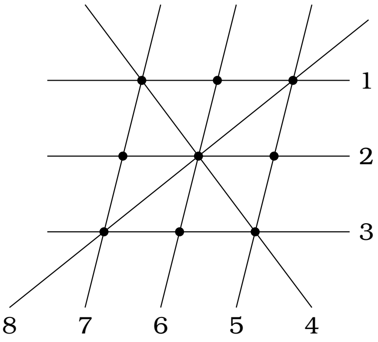
In **construction 2**, we have $n-1$ concurrent lines, and one line which is parallel to one of the concurrent lines and does not pass through the common point of intersection. The last line intersects every concurrent line other than the one it is parallel to, creating $n-2$ points of intersection. Together with the point of intersection of the concurrent lines, we get $n-1$ points on intersection in total.

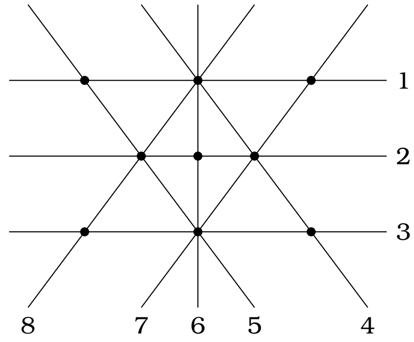
2. n **lines and** n **distinct points of intersection**



Here we have $n-1$ concurrent lines, and one line which is not parallel to any of the concurrent lines and does not pass through the common point of intersection. The last line intersects every concurrent line, creating $n-1$ points of intersection. Together with the point of intersection of the concurrent lines, we get n points on intersection in total.

3. 8 **lines and** 9 **distinct points of intersection**





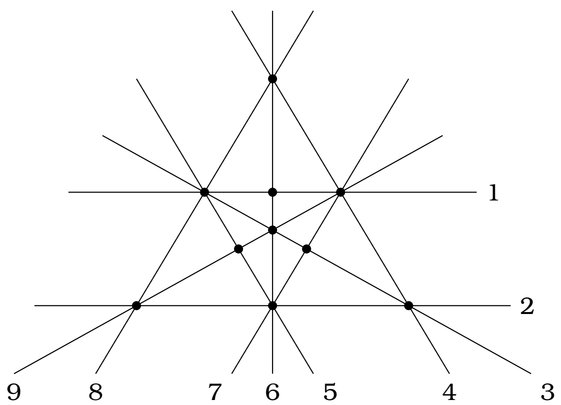
Construction 1 **Construction 2**

Either of these constructions provide us with an example of 8 lines with 9 distinct points of intersection.

In **construction 1**, we take four distinct lines that create a parallelogram (lines $1, 3, 5, 7$ here), the lines forming the diagonals of the parallelogram (lines 4 and 8 here) and the two lines joining the midpoints on the opposite sides of the parallelogram (lines 2 and 6 here). Since this is a parallelogram, lines $2, 4, 6, 8$ are concurrent and meet at the centre of the parallelogram. Also, lines $1, 2, 3$ are parallel and therefore never intersect, same for lines $5, 6, 7$. Rest of the concurrencies follow from the definition of the lines. This confirms the validity of our diagram and that it captured all intersection points of the lines.

In **construction 2**, start with three distinct lines forming a triangle (lines $3, 4, 8$). Then draw three lines joining the midpoints of the sides (lines $2, 5, 7$). Lastly, pick a vertex of the triangle, and draw two lines through it; the median (line 6) and the line parallel to the opposite side to the vertex (line 1). The concurrencies follow from the definition of the lines, and any side of a triangle is parallel to the line joining the midpoints of the other two sides. This confirms that we have captured all intersection points in our diagram.

4. 9 **lines and** 10 **distinct points of intersection**



Here we start with three distinct lines that form a triangle (lines $2, 4, 8$ here). Then draw the the lines joining the midpoints of the sides of the triangle (lines $1, 5, 7$ here), as well as the medians of the triangle (lines $3, 6, 9$ here). The medians of any triangle are concurrent at the centroid. Every other concurrency in the picture follows from definition of the lines. Also, lines 1 and 2 are parallel, so they never intersect, same for the pair of lines 4 and 5 as well as the pair 7 and 8. This validates our diagram and ensures that it captures all points of intersection.

Remark: In this construction, removing any one line other than $2, 4$ or 8 gives an answer to 8 lines with 9 intersection points as well. Since removing any median or any line joining the midpoints of two sides removes exactly 1 intersection point.

□

Problem 2. Let a, b, c be distinct nonzero real numbers satisfying

$$a + \frac{2}{b} = b + \frac{2}{c} = c + \frac{2}{a}.$$

Determine the value of $|a^2b + b^2c + c^2a|$.

Solution 1. We have

$$a + \frac{2}{b} = b + \frac{2}{c} \implies abc + 2c = b^2c + 2b.$$

Similarly, $abc + 2a = c^2a + 2c,\ abc + 2b = a^2b + 2a$. Summing, we get $a^2b + b^2c + c^2a = 3abc$.

On the other hand, we can rewrite $abc + 2c = b^2c + 2b$ as $bc(a-b) = 2(b-c)$ and similarly $ca(b-c) = 2(c-a)$ and $ab(c-a) = 2(a-b)$. Multiplying these and cancelling $(a-b)(b-c)(c-a)$ from both sides (since a, b, c are distinct), we get that $(abc)^2 = 8$. Therefore, $|abc| = 2\sqrt{2}$.

Thus, $|a^2b + b^2c + c^2a| = 3|abc| = 6\sqrt{2}$.

□

Solution 2. We try to solve the equations directly, and find the values of b and c in terms of a. We start by noticing

$$b + \frac{2}{c} = c + \frac{2}{a} \implies b = c + \frac{2}{a} - \frac{2}{c} = \frac{c^2a + 2c - 2a}{ca}.$$

Substituting this value of b in the equation $c + \frac{2}{a} = a + \frac{2}{b}$ and clearing denominators gives us

$$\begin{aligned}
&(2 + ac)(ac^2 + 2c - 2a) = a^2(c^2a + 2c - 2a) + 2a^2c \\
\implies &a^2c^3 - a^3c^2 + 4ac^2 - 6a^2c + 2a^3 + 4c - 4a = 0 \\
\implies &(c-a)(a^2c^2 + 4ac - 2a^2 + 4) = 0 \\
\implies &a^2c^2 + 4ac - (2a^2 - 4) = 0 \\
\implies &c = \frac{-4a \pm \sqrt{16a^2 + 4a^2(2a^2-4)}}{2a^2} = \frac{-4a \pm 2\sqrt{2}a^2}{2a^2} = \frac{-2 \pm \sqrt{2}a}{a}.
\end{aligned}$$

Now, we can obtain the value of b as

$$b = c + \frac{2}{a} - \frac{2}{c} = \frac{-2}{a} \pm \sqrt{2} + \frac{2}{a} - \frac{2a}{-2 \pm \sqrt{2}a} = \frac{\pm(-2\sqrt{2}) + 2a - 2a}{-2 \pm \sqrt{2}a} = \frac{2}{\pm\sqrt{2} - a}$$

Note that, if $a = \sqrt{2}$ or $-\sqrt{2}$, then one of these solutions do not work.
Now, we can calculate the value of

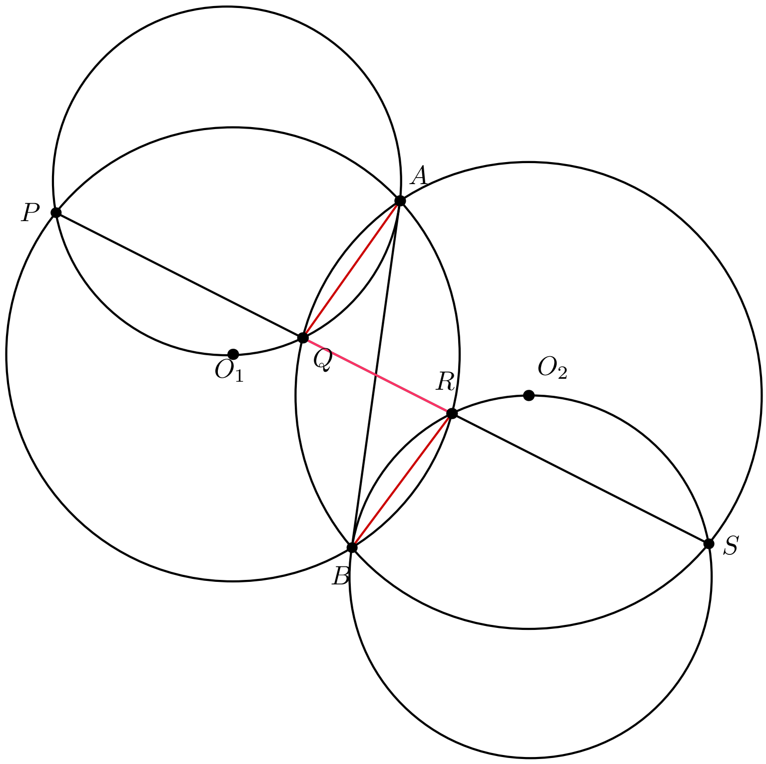
$$\begin{aligned}
|a^2b + b^2c + c^2a| &= \left| \frac{2a^2}{\pm\sqrt{2} - a} + \frac{4\left(-2 \pm \sqrt{2}a\right)}{\left(\pm\sqrt{2} - a\right)^2 a} + \frac{\left(-2 \pm \sqrt{2}a\right)^2 a}{a^2} \right| \\
&= \left| \frac{2a^3 \pm (-4\sqrt{2}) + \left(-2 \pm \sqrt{2}a\right)^2 \left(\pm\sqrt{2} - a\right)}{a\left(\pm\sqrt{2} - a\right)} \right| \\
&= \left| \frac{2a^3 \pm (-4\sqrt{2}) - 2a^3 \pm 4\sqrt{2} \pm 6\sqrt{2}a^2 - 12a}{a\left(\pm\sqrt{2} - a\right)} \right| \\
&= \left| \frac{6\sqrt{2}a(\pm a - \sqrt{2})}{a(\pm\sqrt{2} - a)} \right| = 6\sqrt{2}.
\end{aligned}$$

□

Remark. As observed in solution 2, there are infinitely many distinct non-zero a, b, c which satisfy the assertion. Any triple of the form $\left(x, \frac{-2}{x + \sqrt{2}}, \frac{-\sqrt{2}x - 2}{x}\right)$ satisfies the stated condition where $x \neq -\sqrt{2}$. So in particular for $x = 2\sqrt{2}$ we have $\left(2\sqrt{2}, -\frac{-\sqrt{2}}{3}, \frac{-3}{\sqrt{2}}\right)$.

Problem 3. Let Ω and Γ be circles centred at O_1, O_2 respectively. Suppose that they intersect in distinct points A, B. Suppose O_1 is outside Γ and O_2 is outside Ω. Let ℓ be a line not passing through A and B that intersects Ω at P, R and Γ at Q, S so that P, Q, R, S lie on the line in this order. Furthermore, the points O_1, B lie on one side of ℓ and the points O_2, A lie on the other side of ℓ. Given that the points A, P, Q, O_1 are concyclic and B, R, S, O_2 are concyclic as well, prove that $AQ = BR$.

Solution.



Claim: $AQ = QR = RB$.

Proof. Note that $\angle QAR + \angle ARQ = \angle AQP = \angle AO_1P = 2 \times \angle ARQ$. Thus, we have $\angle QAR = \angle ARQ$, and hence $AQ = QR$.
Similar considerations with $\angle BRS = \angle BO_2S = 2 \times \angle BQS$, gives us that $\angle BQR = \angle RBQ$, hence $QR = RB$. □

Thus, we indeed have $AQ = BR$ as desired. □

Problem 4. Prove there do not exist positive rational numbers x and y such that

$$x + y + \frac{1}{x} + \frac{1}{y} = 2025.$$

Solution 1. For the sake of contradiction, let us assume that the equation has a rational solution $x = \dfrac{p}{q}$ and $y = \dfrac{r}{s}$, where p, q, r, s are positive integers with $\gcd(p, q) = 1 = \gcd(r, s)$. Then, we have

$$\begin{aligned}
&\frac{p}{q} + \frac{q}{p} + \frac{r}{s} + \frac{s}{r} = 2025 \\
\implies &\frac{p^2 + q^2}{pq} + \frac{r^2 + s^2}{rs} = 2025 \\
\implies &rs(p^2 + q^2) + pq(r^2 + s^2) = 2025pqrs.
\end{aligned}$$

This implies that $pq \mid rs(p^2 + q^2)$. Since $\gcd(p, q) = 1$, we have $\gcd(pq, p^2 + q^2) = 1$ [Since $\gcd(p, p^2 + q^2) = \gcd(p, q^2) = 1$ and similarly $\gcd(q, p^2 + q^2) = \gcd(q, p^2) = 1$]. Therefore, $pq \mid rs$. Similarly, we can prove that $rs \mid pq$. Together, we get $pq = rs$ [Since p, q, r, s are all positive, $pq = -rs$ is not a possibility].

Our equation thereby reduces to

$$p^2 + q^2 + r^2 + s^2 = 2025pq.$$

Now, since $pq = rs$ and $\gcd(p,q) = 1 = \gcd(r,s)$, we see that either none of p, q, r, s are divisible by 3, or exactly two of them are. In the first case,

$$p^2 + q^2 + r^2 + s^2 \equiv 1 + 1 + 1 + 1 \equiv 1 \pmod 3.$$

In the second case,

$$p^2 + q^2 + r^2 + s^2 \equiv 0 + 1 + 0 + 1 \equiv 2 \pmod 3.$$

Both cases lead to a contradiction since $3 \mid 2025pq = p^2 + q^2 + r^2 + s^2$.

Therefore, no positive rational solutions exist for the equation $x + y + \frac{1}{x} + \frac{1}{y} = 2025$. $\square$

Solution 2. As in solution 1, we assume that the equation has a rational solution $x = \frac{p}{q}$ and $y = \frac{r}{s}$, where p, q, r, s are positive integers $\gcd(p,q) = 1 = \gcd(r,s)$; then proceed to show that $pq = rs$.

This implies that there exists positive integers a, b, c, d such that

$$p = ab, \quad q = cd, \quad r = ac, \quad s = bd.$$

[To see this, one can take $a = \gcd(p,r)$, $b = \gcd(p,s)$, $c = \gcd(q,r)$, $d = \gcd(q,s)$.]

Our equation reduces to

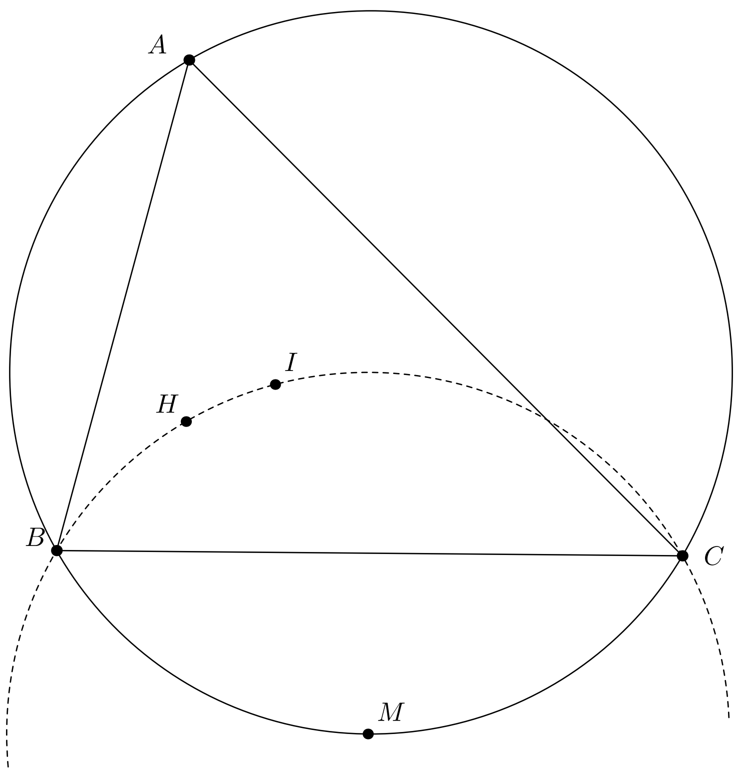
$$2025abcd = a^2b^2 + c^2d^2 + a^2c^2 + b^2d^2 = (a^2 + d^2)(b^2 + c^2).$$

Now, note that a, b, c, d are pairwise coprime [a and b are coprime to c and d as $\gcd(p,q) = 1$, and $\gcd(r,s) = 1$ imples that $\gcd(a,b) = \gcd(c,d) = 1$]. So, at most one of a, b, c, d is divisible by 3. This implies $a^2 + d^2$ is 1 or 2 modulo 3, and the same for $b^2 + c^2$. This leads to a contradiction, since 3 divides $2025abcd$.

Therefore, no positive rational solutions exist for the equation $x + y + \frac{1}{x} + \frac{1}{y} = 2025$. $\square$

Problem 5. Let ABC be an acute-angled triangle with $AB < AC$, orthocenter H and circumcircle Ω. Let M be the midpoint of minor arc BC of Ω. Suppose that MH is equal to the radius of Ω. Prove that $\angle BAC = 60^\circ$.

Solution 1.



Let I be the incenter of $\triangle ABC$, and let the circumradius of $\triangle ABC$ be R. Note that the circumradius of $\triangle BHC$ is also R: To see this, one may use sin rule

$$R_{BHC} = \frac{BC}{2\sin\angle BHC} = \frac{BC}{2\sin(180^\circ - \angle BAC)} = \frac{BC}{2\sin\angle BAC} = R,$$

or else one may note that the circumcircle of $\triangle BHC$ is the reflection of the circumcircle of $\triangle ABC$ about BC (since the reflection of H across BC lies on Ω).

M is the unique point satisfying $MH = R$, $MB = MC$ and **(*)** *M lies on opposite side of BC as H* (this is because H lies in the interior of the triangle whereas M lies in the exterior), so it must be the circumcenter of $\triangle BHC$.

Now M is also the circumcenter of $\triangle BIC$ by the Incenter-Excenter lemma, so we have B, H, I, C cyclic with center M. Thus, $180^\circ - \angle BAC = \angle BHC = \angle BIC = 90^\circ + \angle BAC/2$, giving that $\angle BAC = 60^\circ$ as needed. $\square$

Solution 2. Consider the reflections of H, M across BC, let us call them H', M' respectively. Then it is well known that H' lies on Ω.

Now, M' is the unique point lying on the perpendicular bisector of BC, with $M'H' = R$, and **(*)** *which is on the opposite side of BC as H'*, thus M' has to be the center of Ω.

Finally, we can conclude that $180^\circ - \angle BAC = \angle BMC = \angle BM'C = 2\angle BAC$, leading to the conclusion that $\angle BAC = 60^\circ$. $\square$

Solution 3. Let O be the center of Ω, and consider the quadrilateral $AOMH$. We have $AO = OM = MH = R$, $\angle HAM = \angle MAO$ and $OM \parallel AH$. This is not alone enough to conclude that $AOMH$ is a rhombus (see remark), but combining it with the fact that $\angle AOM$ and $\angle AHM$ are both obtuse we can conclude as follows: sine rule in $\triangle AOM$ and $\triangle AHM$ gives us

$$\frac{AM}{\sin\angle AHM} = \frac{MH}{\sin\angle HAM} = \frac{R}{\sin\angle HAM} = \frac{R}{\sin\angle MAO} = \frac{AM}{\sin\angle AOM}$$

Now, **(*)** *both $\angle AOM$ and $\angle AHM$ being obtuse* allows us to conclude that they are in fact equal, which means that $\triangle AOM \cong \triangle AHM$ making $\square AOHM$ a rhombus. From here, one can conclude by noting that this means $AH = R$ so $2R\cos A = R$, hence $\angle BAC = 60^\circ$. $\square$

Remark: The assumption that $\triangle ABC$ is acute is crucial: it is easy to provide a counterexample where $MH = R$ but $\triangle ABC$ is obtuse with $\angle BAC \neq 60^\circ$. The places we have used this assumption in each solution has been marked with a **(*)**.

Problem 6. Let $p(x)$ be a nonconstant polynomial with integer coefficients, and let $n \geq 2$ be an integer such that no term of the sequence

$$p(0),\ p(p(0)),\ p(p(p(0))), \ldots$$

is divisible by n. Show that there exist integers a, b such that $0 \leq a < b \leq n-1$ and n divides $p(b) - p(a)$.

Solution 1. Given any function $f : S \to S$ and a natural number m, we define $f^m : S \to S$ to be the composition of f with itself m times. Note that the integer valued polynomial $p(x)$ defines a function $p : \mathbb{N} \to \mathbb{N}$.

Let us assume that there are no integers $0 \leq a < b \leq n-1$ such that $n \mid (p(b) - p(a))$. Then $p(0), p(1), \ldots, p(n-1)$ all have distinct remainders when divided by n. Define the set $T = \{0, 1, \ldots, n-1\}$, and the map $g : T \to T$ where $g(a)$ is the remainder when $p(a)$ is divided by n. Since g is injective and T is finite, g is a permutation.

Let $(0, a_1, \ldots, a_{k-1})$ be the cycle containing 0 in the permutation g. Then $g^k(0) = 0$.

Finally, since $p(x)$ has integer coefficients, if $a \equiv b \pmod n$, then $p(a) \equiv p(b) \pmod n$. So,

Lemma. For all $m \in \mathbb{N}$, we have $p^m(0) \equiv g^m(0) \pmod{n}$.

Proof. We proceed by induction on m. The base case of $m = 1$ follows from the definition of g.

For the induction step, assume that $p^m(0) \equiv g^m(0) \pmod{n}$. Therefore,

$$p^{m+1}(0) = p(p^m(0)) \equiv p(g^m(0)) \equiv g(g^m(0)) \equiv g^{m+1}(0) \pmod{n}.$$

This completes the induction step. Therefore the statement holds for all $m \in \mathbb{N}$. $\square$

So, we get $p^k(0) \equiv g^k(0) \equiv 0 \pmod{n}$. This contradicts the given condition that $p^k(0)$ is not divisible by n for any k. Therefore, our initial assumption was false, i.e. there exist integers $0 \leq a < b \leq n-1$ such that $n \mid (p(b) - p(a))$. $\square$

Solution 2. Given any function $f : S \to S$ and a natural number m, we define $f^m : S \to S$ to be the composition of f with itself m times. Note that the integer valued polynomial $p(x)$ defines a function $p : \mathbb{N} \to \mathbb{N}$.

By pigeonhole principle, we see that two of the first $n+1$ terms of the sequence have the same remainder modulo n. That is, there exists $1 \leq k < \ell \leq n+1$ such that $p^\ell(0) \equiv p^k(0) \pmod{n}$.

Let us assume that there are no integers $0 \leq a < b \leq n-1$ such that $n \mid (p(b) - p(a))$. Now, since p is integer valued, if $n \mid b - a$, then $n \mid (p(b) - p(a))$. Therefore, p defines a function from $T = \{0, 1, \ldots, n-1\}$, the set of residues modulo n to itself. This map is injective.

So, we get that if $p(b) \equiv p(a) \pmod{n}$, then $b \equiv a \pmod{n}$. By using this result inductively, we can prove that for all positive integers m, if $p^m(b) \equiv p^m(a) \pmod{n}$, then $b \equiv a \pmod{n}$.

Since we have $p^\ell(0) \equiv p^k(0) \pmod{n}$, we get $p^{\ell-k}(0) \equiv 0 \pmod{n}$. This contradicts the given condition that $p^k(0)$ is not divisible by n for any k. Therefore, our initial assumption was false, i.e. there exist integers $0 \leq a < b \leq n-1$ such that $n \mid (p(b) - p(a))$. $\square$