KV RMO 2025

Official Solutions

Problem 1. Solve the following system of equations in nonnegative integers $a_1, a_2, \dots, a_8$ where $a_i \neq 1$ for $i = 1, \dots, 8$:

$$\begin{aligned} a_1a_2 &= a_3 + a_4, \\ a_3a_4 &= a_5 + a_6, \\ a_5a_6 &= a_7 + a_8, \\ a_7a_8 &= a_1 + a_2. \end{aligned}$$

Solution 1. First we deal with the case where some $a_i = 0$. Since the equations are cyclic in nature, we can assume without loss of generality that $a_8 = 0$. Then, $a_1 + a_2 = 0$, which implies $a_1 = a_2 = 0$. Now, $a_3 + a_4 = 0 \cdot 0 = 0$, leading to $a_3 = a_4 = 0$. Similarly , we get $a_5 + a_6 = 0$, which means $a_5 = a_6 = 0$. And lastly $a_7 = a_7 + a_8 = 0 \cdot 0 = 0$. Observe that $a_i = 0$ for all i is a valid solution for this system as $0 \cdot 0 = 0 + 0$.

Now, let us come to the case where $a_i \geq 2$ are positive integers. Then for all $1 \leq k \leq 4$, if $b_k = \max(a_{2k-1}, a_{2k})$, we have

$$a_{2k-1}a_{2k} \geq 2b_k \geq a_{2k-1} + a_{2k}.$$

Note that equality implies $\min(a_{2k-1}, a_{2k}) = 2$ and $b_k = a_{2k-1} = a_{2k}$; therefore $a_{2k-1} = a_{2k} = 2$.

We now have a chain of inequalities

$$a_1 + a_2 \leq a_1a_2 = a_3 + a_4 \leq a_3a_4 = a_5 + a_6 \leq a_5a_6 = a_7 + a_8 \leq a_7a_8 = a_1 + a_2.$$

Therefore, every inequality must be an equality, Which implies $a_i = 2$ for all i. Since $2 \cdot 2 = 2 + 2$, we can see that $a_i = 2$ for all i is a solution.

So, the only solutions are $a_i = 0$ for all $1 \leq i \leq 8$, or $a_i = 2$ for all $1 \leq i \leq 8$.

□

Solution 2. As in solution 1, we deal with the case where some $a_i = 0$ at first.

Adding all the equations, we get

$$\sum_{k=1}^{4} a_{2k-1}a_{2k} = \sum_{k=1}^{4} a_{2k-1} + a_{2k}.$$

Using the identity $ab - a - b - 1 = (a-1)(b-1)$, we get that

$$\sum_{k=1}^{4} (a_{2k-1} - 1)(a_{2k} - 1) = 4.$$

Since $a_i \geq 2$ for all i, we have $(a_{2k-1} - 1)(a_{2k} - 1) \geq 1$ for all k. Therefore, these must all be equalities. Which implies that $a_i = 2$ for all i. Since $2 \cdot 2 = 2 + 2$, we can see that $a_i = 2$ for all i is a solution.

So, the only solutions are $a_i = 0$ for all $1 \leq i \leq 8$, or $a_i = 2$ for all $1 \leq i \leq 8$. □

Problem 2. Let a, b, c be positive real numbers satisfying $abc = 1$. Prove that

$$\frac{2a^2}{a^3+1} + \frac{2b^2}{b^3+1} + \frac{2c^2}{c^3+1} \leq a^2 + b^2 + c^2.$$

Solution 1. By AM-GM inequality, $a^3+1 \geq 2a\sqrt{a}$, $b^3+1 \geq 2b\sqrt{b}$, $c^3+1 \geq 2c\sqrt{c}$. Therefore,

$$\frac{2a^2}{a^3+1}+\frac{2b^2}{b^3+1}+\frac{2c^2}{c^3+1} \leq \frac{2a^2}{2a\sqrt{a}}+\frac{2b^2}{2b\sqrt{b}}+\frac{2c^2}{2c\sqrt{c}} = \sqrt{a}+\sqrt{b}+\sqrt{c}. \tag{1}$$

Moreover, $a^2+b^2+c^2-ab-bc-ca = \frac{1}{2}\left((a-b)^2+(b-c)^2+(c-a)^2\right) \geq 0$. This implies,

$$a^2+b^2+c^2 \geq ab+bc+ca = \frac{ab+bc+ca}{abc} = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}. \tag{2}$$

Using the same inequality with a replaced by $\frac{1}{\sqrt{a}}$ etc, we get

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}} = \sqrt{c}+\sqrt{a}+\sqrt{b}. \tag{3}$$

Combining equations (1), (2) and (3), we are done. □

Solution 2. We will prove that

$$\frac{a^2(a^3-1)}{a^3+1} \geq \frac{3}{2}(a-1).$$

To prove this, note that

$$\frac{a^2(a^3-1)}{a^3+1} - \frac{3}{2}(a-1) = \frac{(a-1)^2(2a^3+a^2+3a+3)}{2(a^3+1)} \geq 0.$$

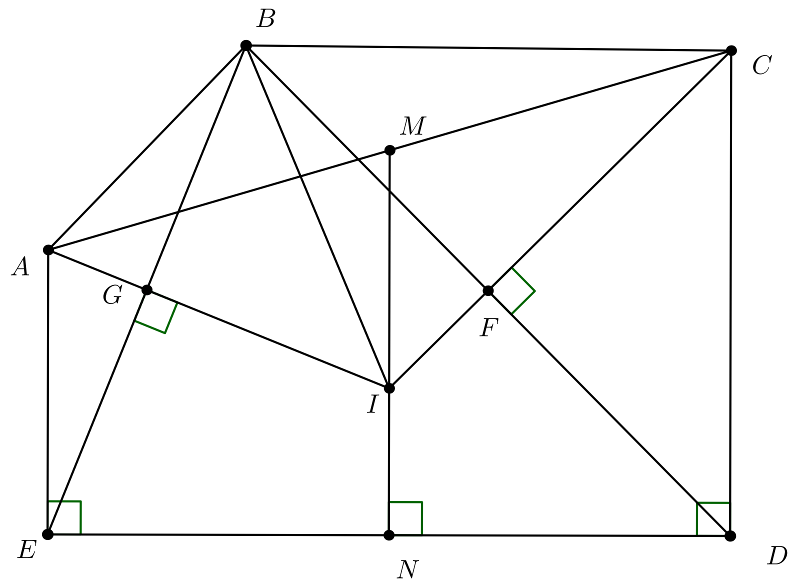
Thus, we have

$$(a^2+b^2+c^2)-\left(\frac{2a^2}{a^3+1}+\frac{2b^2}{b^3+1}+\frac{2c^2}{c^3+1}\right) = \frac{a^2(a^3-1)}{a^3+1}+\frac{b^2(b^3-1)}{b^3+1}+\frac{c^2(c^3-1)}{c^3+1} \geq \frac{3}{2}(a+b+c-3).$$

Now, it suffices to note that $a+b+c \geq 3\sqrt[3]{abc} = 3$ by AM-GM inequality. □

Problem 3. Let $ABCDE$ be a convex pentagon in which $AB = AE$, $CB = CD$, and $\angle AED = \angle CDE = 90°$. Let the internal bisectors of $\angle EAB$ and $\angle DCB$ intersect at I, and let M be the midpoint of AC. Prove that $\angle MIC = \angle AIB$.

Solution.



Let the midpoints of BE, ED, DB be G, N, F respectively.
Claim: I is the circumcenter of $\triangle BED$.

Proof. Note that $AB = AE$ implies that the internal bisector of $\angle EAB$ is the same as the perpendicular bisector of BE. Similarly, $CB = CD$ implies that the internal bisector of $\angle DCB$ is the same as the perpendicular bisector of BD. Hence, their point of intersection must be the circumcenter of $\triangle BED$. □

Now, note that $AEDC$ is a trapezium with $AE \parallel CD$ and $AE \perp ED$. Thus, the perpendicular bisector IN of DE is also parallel to AE and CD, and hence passes through M. Finally, we now have

$$\begin{aligned}
\angle MIC &= \angle ICD && (MI \parallel CD)\\
&= 90^\circ - \angle CDF\\
&= \angle BDE && (\angle CDE = 90^\circ)\\
&= \angle BIG && (I \text{ is the circumcenter of } \triangle BDE)\\
&= \angle BIA.
\end{aligned}$$

This completes the solution. □

Problem 4. A frog is initially at $(0,0)$ and it reaches $(n,2)$, $n \geq 1$, using the following moves in any order several times:

(i) $R = (1,0)$, that is, if the frog is at (a,b) it goes to $(a+1,b)$;

(ii) $U = (0,1)$, that is, if the frog is at (a,b) it goes to $(a,b+1)$;

(iii) $D = (1,1)$, that is, if the frog is at (a,b) it goes to $(a+1,b+1)$.

In how many ways can the frog go from $(0,0)$ to $(n,2)$, $n \geq 1$, using the above steps subject to the condition that steps of the type UU, DD are forbidden?
(For example, for $n = 3$, $RDUR, DRD$ are admissible paths, while $DDR, RUURR$ are not.)

Solution 1. Any such path consists of two climb moves (D or U) and rest Rs. Breaking into cases:

- **Both U:** This has n R moves and two U moves. Since we are not allowed to have UU, if we line up the Rs first, then there are $n+1$ gaps, and two of which must be filled by the U moves. So there are $\binom{n+1}{2}$ such paths.
- **Both D:** This has $n-2$ R moves, and two D moves. Since we are not allowed to have DD, like the last case, there are $\binom{n-1}{2}$ such paths.
- **U and D:** This has $n-1$ many R moves, one D move, and one U move. There are no restrictions this time. So, out of $n+1$ moves, we can select the U move in $n+1$ ways, and then the D move in n ways, and the rest are now Rs. Therefore, there are $(n+1)n$ such paths.

So, the total number of paths of this type is

$$\binom{n+1}{2} + \binom{n-1}{2} + n(n+1) = \frac{1}{2}(n(n+1) + (n-1)(n-2) + 2n(n+1)) = 2n^2 + 1.$$

□

Solution 2. Let's first count paths from $(0,0)$ to $(n,2)$ consisting of R, D and U. Any such path must have two climb moves (D or U) and the others as R. Breaking into cases:

- **Both D:** This has $n-2$ R moves, and two D moves: so there are $\binom{n}{2}$ such paths.
- **Both U:** This has n R moves, and two U moves: so there are $\binom{n+2}{2}$ such paths.
- **U and D:** This has $n-1$ many R moves, one D move, and one U move: so there are $(n+1)n$ such paths.

So, the number of total paths is $\binom{n}{2} + \binom{n+2}{2} + n(n+1)$.

Now, the number of paths with a DD is equal to $n-1$. The number of paths with a UU is equal to $n+1$.

So, number of paths satisfying the restrictions in the question is equal to

$$\binom{n+2}{2} + \binom{n}{2} + n(n+1) - 2n = \frac{1}{2}(n^2 + 3n + 2 + n^2 - n + 2n^2 + 2n - 4n) = 2n^2 + 1.$$

□

Solution 3. Let the number of such paths be a_n, we call these valid paths. Then, $a_1 = 3$ (UD, DU and URU) and $a_2 = 9$ (all permutations of DRU, and $RURU$, $URRU$, $URUR$). We want to find a recursive relation for a_n.

Let $n \geq 3$, and let the set of valid paths ending in R be S_n. Clearly, $|S_n| = a_{n-1}$. Let the set of valid paths starting with R be T_n. It is easy to see that $|T_n| = a_{n-1}$, since T_n is in bijection with number of valid paths from $(1, 0)$ to $(n, 2)$. Also $S_n \cap T_n$ is in bijection with valid paths from $(1, 0)$ to $(n-1, 2)$, so $|S_n \cap T_n| = a_{n-2}$. So, $|S_n \cup T_n| = 2a_{n-1} - a_{n-2}$.

Lastly, we count valid paths that belong to neither S_n nor T_n. Note that such a path starts with U or D, and ends with U or D as well. So, the rest of the path has to consist of Rs only. So, number of such paths is equal to 4.

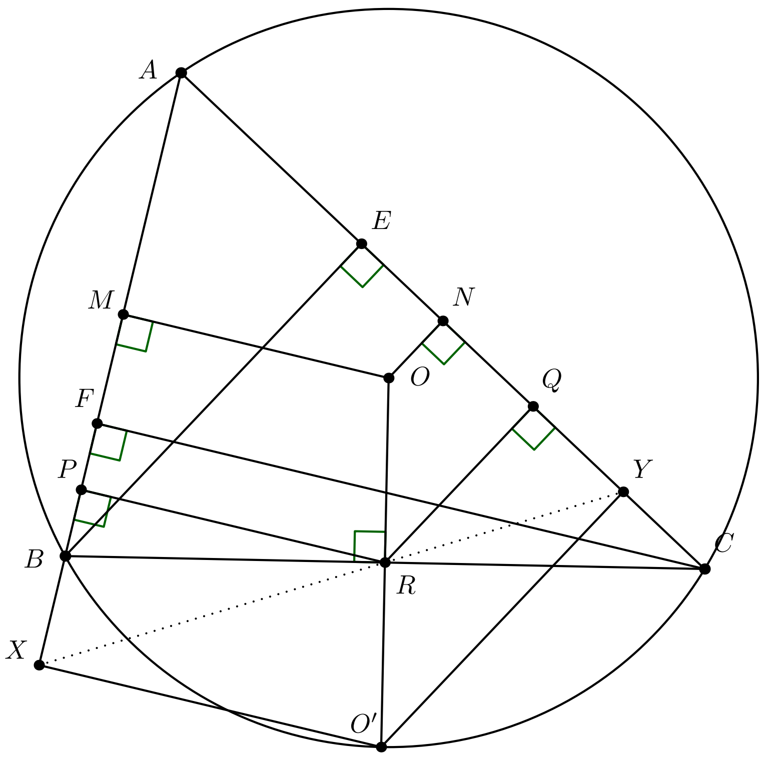
This gives us the recursive relation $a_n = 2a_{n-1} - a_{n-2} + 4$. To resolve this, define $b_n = a_n - a_{n-1}$ for all $n \geq 2$. Then $b_2 = 6$ and $b_n = b_{n-1} + 4$, which implies $b_n = b_2 + 4(n-2) = 4n - 2$ for all $n \geq 2$. Thus,

$$a_n = a_1 + \sum_{k=2}^{n} b_k = 1 + \sum_{k=1}^{n} (4k-2) = 1 + 2n^2.$$

□

Problem 5. Let ABC be an acute-angled triangle with $\angle BAC = 60^\circ$ and $AB < BC < AC$. Let M, N be the midpoints of AB, AC respectively. Suppose BE, CF are altitudes, with E on CA and F on AB. Let X be the image of M under reflection in the midpoint of BF, and Y be the image of N under reflection in the midpoint of CE. Prove that XY bisects BC.

Solution.



Let the midpoint of BF be P and the midpoint of CE be Q. Let R be the midpoint of BC.

Let O' be the reflection of the circumcenter across BC, then note that $\angle BO'C = \angle BOC = 2 \times 60^\circ = 120^\circ$, so O' lies on the circumcircle of $\triangle ABC$.

Claim: XY is the Simson line of O' with respect to $\triangle ABC$.

Proof. It suffices to show that X is the foot of perpendicular from O' onto AB. (Y being the foot of perpendicular from O' onto AC will then follow by a similar argument)

Note that the foot of perpendicular from O onto AB is M, the foot of perpendicular from R onto AB is P (by Midpoint theorem on $\triangle BFC$) and so the foot of perpendicular from O' onto AB is indeed the image of M in P, which is X as desired. □

Now, note that R is the foot of perpendicular of O' onto BC, thus the Simson line XY of O' w.r.t. $\triangle ABC$ indeed passes through R as desired. □

Problem 6. Define the sequence $\langle a_0, a_1, a_2, \ldots \rangle$ as follows: $a_0 = 49$ and $a_n = 10^{2^n} a_{n-1} - 1$ for $n \geq 1$. Show that $s(a_n^2) = n^2 + n + 7$ for all $n \geq 0$, where $s(m)$ denotes the sum of digits in base 10 representation of a nonnegative integer m.

Solution. We begin by establishing some basic properties. Let $b_n = 2a_n$ for all n. Then $b_0 = 98$, and $b_n = 10^{2^n} b_{n-1} - 2$ for $n \geq 1$. Whenever we speak of digits or length of a number in this solution, it is always in the base 10 representation.

Lemma. For all $n \geq 1$, the following properties hold:

1. The last digit of a_n^2 is 1.
2. The last digit of b_n is 8.
3. The length of b_n is 2^{n+1}.
4. The digits of b_n are: n many 7s, one 8 at the end, and the rest 9s.

Proof. 1. We see that $a_0^2 = 2401$. Now, for any $n \geq 1$, we have $a_n = 10^{2^n} a_{n-1} - 1 \equiv 9 \pmod{10}$, therefore, $a_n^2 \equiv 9^2 \equiv 1 \pmod{10}$.

2. Again, $b_0 = 98$, and for any $n \geq 1$, we have $b_n = 10^{2^n} b_{n-1} - 2 \equiv 8 \pmod{10}$.

3. Let l_n be the length of b_n. We know that $l_0 = 2$. Since $b_n = 10^{2^n} b_{n-1} - 2$, we can see that $l_n = l_{n-1} + 2^n$. [As $10^{2^n} b_{n-1}$ ends with $800 \ldots 00$, and subtracting 2 from $800 \ldots 00$ creates $799 \ldots 98$.]

   Therefore $l_n = l_0 + \sum_{k=1}^{n} 2^k = 2 + 2^{n+1} - 2 = 2^{n+1}$.

4. We use induction on n. The base case follows as $b_0 = 98$.

   For induction step, let us assume the result for $n - 1$, i.e. b_{n-1} consists of $n - 1$ many 7s, one 8 at the end, and the rest 9s. Now, $10^{2^n} b_{n-1}$ is digits of b_{n-1} followed by 2^n many 0s, so it ends in $800 \ldots 00$. As earlier, subtracting 2 from $800 \ldots 00$ creates $799 \ldots 98$. So, we see that b_n also consists of only 7,8 and 9s, the only occurrence of 8 is the singular one at the end, and the number of 7s is one more than the number of 7s in b_{n-1}, i.e. n. This completes our induction step.

□

Finally, we apply induction to prove our main result, that $s(a_n^2) = n^2 + n + 7$. Note that $s(a_0^2) = s(2401) = 7 = 0^2 + 0 + 7$, which provides the base case. For the induction step, assume that the result holds for $n - 1$, that is, $s(a_{n-1}^2) = (n-1)^2 + (n-1) + 7 = n^2 - n + 7$.

Now, $a_n^2 = \left(10^{2^n} a_{n-1} - 1\right)^2 = 10^{2^{n+1}} a_{n-1}^2 - 2 \cdot 10^{2^n} a_{n-1} + 1 = 10^{2^n} \left(10^{2^n} a_{n-1}^2 - b_{n-1}\right) + 1$. Therefore,

$$s(a_n^2) = 1 + s(10^{2^n} a_{n-1}^2 - b_{n-1})$$

Now, $10^{2^n} a_{n-1}^2$ is just digits of a_{n-1}^2 followed by 2^n many 0s. Also, size of b_{n-1} is 2^n, so $b_{n-1} < 10^{2^n}$. Therefore $10^{2^n} a_{n-1}^2 - b_{n-1} = 10^{2^n} (a_{n-1}^2 - 1) + \left(10^{2^n} - b_{n-1}\right)$ looks like: digits of $a_{n-1}^2 - 1$ followed by digits of $10^{2^n} - b_{n-1}$. Recall that a_{n-1}^2 ends in 1, so $s(a_{n-1}^2 - 1) = s(a_{n-1}^2) - 1$. Therefore,

$$s(10^{2^n} a_{n-1}^2 - b_{n-1}) = s(a_{n-1}^2 - 1) + s(10^{2^n} - b_{n-1}) = n^2 - n + 6 + s(10^{2^n} - b_{n-1}).$$

Finally, $10^{2^n} - 1 = 99 \ldots 9$, so while subtracting any number of size 2^n from $10^{2^n} - 1$, there are no carry overs, which gives

$$s(10^{2^n} - b_{n-1}) = s(10^{2^n} - 1) - s(b_{n-1} - 1) = 9 \cdot 2^n - 9(2^n - n) - 7n = 2n,$$

since $b_{n-1} - 1$ consists of n many 7s, and rest 9s. This leads to

$$s(a_n^2) = 1 + s(10^{2^n} a_{n-1}^2 - b_{n-1}) = 1 + n^2 - n + 6 + s(10^{2^n} - b_{n-1}) = n^2 - n + 7 + 2n.$$

This completes the induction step. □