INMO 2023

Official Solutions

Problem 1. Let S be a finite set of positive integers. Assume that there are precisely 2023 ordered pairs (x, y) in $S \times S$ so that the product xy is a perfect square. Prove that one can find at least four distinct elements in S so that none of their pairwise products is a perfect square.

Note: As an example, if $S = \{1, 2, 4\}$, there are exactly five such ordered pairs: $(1, 1)$, $(1, 4)$, $(2, 2)$, $(4, 1)$, and $(4, 4)$.

Solution. Consider the graph whose vertices are elements of S, with an edge between x and y if and only if xy is a perfect square. We claim every connected component is a clique.

Indeed, take any two vertices corresponding to x, y in S in the same connected component. It suffices to show they are adjacent. By assumption, there is a path between them; so there is a sequence $x = a_1, a_2, \ldots, a_{n-1}, a_n = y$ so that $a_i a_{i+1}$ is a perfect square for $1 \leq i < n$. Therefore

$$xy = a_1 a_n = \frac{(a_1 a_2)(a_2 a_3) \cdots (a_{n-1} a_n)}{a_2^2 \cdots a_{n-1}^2},$$

is a perfect square as well. This proves our claim.

Now suppose first there are at most 3 connected components, with sizes a, b, c (possibly zero). Note that for $(x, y) \in S \times S$, xy is a perfect square if and only if x, y are in the same component, which can be chosen is $a^2 + b^2 + c^2$ ways. Thus

$$a^2 + b^2 + c^2 = 2023.$$

But since squares can only be $0, 1$ or 4 mod 8, and 2023 is 7 mod 8, the above equation is impossible. Thus our graph must have at least four components. Picking a number from each component, we can now satisfy the requirements of the problem. □

Alternative solution. For a in S, let $S_a = \{x \in S | ax \text{ is a square}\}$. Let a, b be elements of S. Suppose that x is in $S_a \cap S_b$. Then ax and bx are squares and hence $ab = \frac{ax \cdot bx}{x^2}$ is a square. Then for any y in S such that ay is a square it follows that by is a square, so $S_a = S_b$. Hence for two elements a, b in S, either $S_a = S_b$ or $S_a \cap S_b = \emptyset$.

Now, $S = \cup S_a$ where the union runs over elements of S (since $a \in S_a$ for any $a \in S$). Let $S = S_{a_1} \cup S_{a_2} \cup \cdots \cup S_{a_n}$ for some elements $a_1, a_2, \ldots, a_n$ of S such that $S_{a_i} \cap S_{a_j} = \emptyset$ for $1 \leq i < j \leq n$. Then the number of distinct pairs (x, y) of $S \times S$ such that xy is a square is precisely $|S_{a_1}|^2 + |S_{a_2}|^2 + \cdots + |S_{a_n}|^2$. Since $2023 \equiv 7 \pmod 8$ it follows that $n > 3$ as in the previous solution. Thus we have four elements a_1, a_2, a_3, a_4 none of whose pairwise products is a square. □

Problem 2. Suppose $a_0, \ldots, a_{100}$ are positive reals. Consider the following polynomial for each k in $\{0, 1, \ldots, 100\}$:

$$a_{100+k} x^{100} + 100 a_{99+k} x^{99} + a_{98+k} x^{98} + a_{97+k} x^{97} + \cdots + a_{2+k} x^2 + a_{1+k} x + a_k,$$

where indices are taken modulo 101, *i.e.*, $a_{100+i} = a_{i-1}$ for any i in $\{1, 2, \ldots, 100\}$. Show that it is impossible that each of these 101 polynomials has all its roots real.

Solution. Let $n = 50$. For the sake of contradiction, assume that each of these polynomials has all real roots; these roots must be negative. Let

$$-\alpha_{1,k}, -\alpha_{2,k}, \ldots, -\alpha_{2n,k}$$

be the roots of the polynomial

$$a_{2n+k}x^{2n} + 2na_{2n-1+k}x^{2n-1} + a_{2n-2+k}x^{2n-2} + a_{2n-3+k}x^{2n-3} + \cdots + a_{2+k}x^2 + a_{1+k}x + a_k.$$

Indices are taken modulo $2n+1$, so $a_{2n+k} = a_{k-1}$ and $a_{2n-1+k} = a_{k-2}$. Then

$$\sum_{j=1}^{2n} \alpha_{j,k} = 2n.\left(\frac{a_{k-2}}{a_{k-1}}\right);\ \prod_{j=1}^{2n} \alpha_{j,k} = \frac{a_k}{a_{k-1}}.$$

Since the $\alpha_{j,k}$'s are positive, AM-GM inequality can be applied and by virtue of it we are led to

$$\left(\frac{a_{k-2}}{a_{k-1}}\right)^{2n} \geq \frac{a_k}{a_{k-1}}$$

for each k. As both sides of the inequalities are positive, multiplying them we obtain

$$\prod_{k=0}^{2n}\left(\frac{a_{k-2}}{a_{k-1}}\right)^{2n} \geq \prod_{k=0}^{2n}\frac{a_k}{a_{k-1}}.$$

But both sides are equal to 1. Therefore all the $2n+1$ A.M-G.M inequalities are equalities implying that for each k,

$$\alpha_{1,k} = \alpha_{2,k} = \ldots = \alpha_{2n,k} = \frac{a_{k-2}}{a_{k-1}}.$$

Since $n \geq 2$, using Vieta's relations gives

$$\frac{a_{k-3}}{a_{k-1}} = \sum_{1\leq i<j\leq 2n} \alpha_{i,k}\alpha_{j,k} = \binom{2n}{2}\left(\frac{a_{k-2}}{a_{k-1}}\right)^2.$$

Simplifying leads

$$\binom{2n}{2}a_{k-2}^2 = a_{k-1}a_{k-3}$$

for each k. Multiplying all these equations yields

$$\left(\binom{2n}{2}^{2n+1} - 1\right)\left(\prod_{k=0}^{2n} a_k\right)^2 = 0,$$

which shows that at least one $a_k = 0$, a contradiction. □

Alternative solution. As above, one proves that

$$\alpha_{1,k} = \alpha_{2,k} = \ldots = \alpha_{2n,k} = \frac{a_{k-2}}{a_{k-1}}.$$

This implies

$$a_{2n+k}x^{2n} + 2na_{2n-1+k}x^{2n-1} + \cdots + a_{1+k}x + a_k = a_{2n+k}\left(x + \frac{a_{k-2}}{a_{k-1}}\right)^{2n}.$$

For $n \geq 2$, comparing coefficients of x^0 and x^1, we see that

$$a_k = a_{2n+k}\left(\frac{a_{k-2}}{a_{k-1}}\right)^{2n}, \quad a_{k+1} = a_{2n+k}\cdot 2n\left(\frac{a_{k-2}}{a_{k-1}}\right)^{2n-1},$$

whence we obtain

$$\frac{a_{k+1}}{a_k} = 2n\cdot\frac{a_{k-1}}{a_{k-2}}.$$

This must hold for all k. However, if we pick k is such that $\frac{a_{k+1}}{a_k}$ is minimal, we must necessarily have

$$\frac{a_{k+1}}{a_k} \leq \frac{a_{k-1}}{a_{k-2}} < 2n\cdot\frac{a_{k-1}}{a_{k-2}},$$

a contradiction. □

Problem 3. Let $\mathbb{N}$ denote the set of all positive integers. Find all real numbers c for which there exists a function $f : \mathbb{N} \to \mathbb{N}$ satisfying:

(a) for any $x, a \in \mathbb{N}$, the quantity $\frac{f(x+a)-f(x)}{a}$ is an integer if and only if $a = 1$;

(b) for all $x \in \mathbb{N}$, we have $|f(x) - cx| < 2023$.

Solution. We claim that the only possible values of c are $k + \frac{1}{2}$ for some non-negative integer k. The fact that these values are possible is seen from the function $f(x) = \left\lfloor \left(k + \frac{1}{2}\right) x \right\rfloor + 1 = kx + \left\lfloor \frac{x}{2} \right\rfloor + 1$. Indeed, if you have any $x, a \in \mathbb{N}$, then

$$\frac{f(x+a) - f(x)}{a} = \frac{1}{a}\left(ka + \left\lfloor \frac{x+a}{2} \right\rfloor - \left\lfloor \frac{x}{2} \right\rfloor\right) = k + \frac{1}{a}\left(\left\lfloor \frac{x+a}{2} \right\rfloor - \left\lfloor \frac{x}{2} \right\rfloor\right).$$

This is clearly an integer for $a = 1$. But for $a \geq 2$, we have

$$\left\lfloor \frac{x+a}{2} \right\rfloor - \left\lfloor \frac{x}{2} \right\rfloor \geq \left\lfloor \frac{x+2}{2} \right\rfloor - \left\lfloor \frac{x}{2} \right\rfloor = 1.$$

If $a = 2k$, then

$$\left\lfloor \frac{x+a}{2} \right\rfloor - \left\lfloor \frac{x}{2} \right\rfloor = k < 2k = a,$$

and if $a = 2k + 1$ for $k \geq 1$, then

$$\left\lfloor \frac{x+a}{2} \right\rfloor - \left\lfloor \frac{x}{2} \right\rfloor \leq \left\lfloor \frac{x+2k+2}{2} \right\rfloor - \left\lfloor \frac{x}{2} \right\rfloor = k + 1 < 2k + 1 = a.$$

So in either case, the quantity $\left\lfloor \frac{x+a}{2} \right\rfloor - \left\lfloor \frac{x}{2} \right\rfloor$ is strictly between 0 and a, and thus cannot be divisible by a. Thus condition (a) holds; condition (b) is obviously true.

Now let us show these are the only possible values, under the weaker assumption that there exists some $d \in \mathbb{N}$ so that $|f(x) - cx| < d$. It is clear that $c \geq 0$: if $d < f(x) - cx < d$ and $c < 0$, then for large x the range $[cx - d, cx + d]$ consists only of negative numbers and cannot contain $f(x)$.

Now we claim that $c \geq \frac{1}{2}$. Indeed, suppose that $0 \leq c < \frac{1}{2}$, and that $d > 0$ is such that $|f(x) - cx| \leq d$. Pick $N > \frac{2d}{1-2c}$ so that $2(cN + d) < N$. Then the N values $\{f(1), \ldots, f(N)\}$ must be all be in the range $\{1, \ldots, cN + d\}$, and by pigeonhole principle, some three values $f(i), f(j), f(k)$ must be equal. Some two of i, j, k are not consecutive: suppose WLOG $i > j + 1$. Then $\frac{f(i)-f(j)}{i-j} = 0$, which contradicts condition (a) for $x = j$ and $a = i - j$.

Now for the general case, suppose $c = k + \lambda$, where $k \in \mathbb{Z}$ and $\lambda \in [0, 1)$. Let $d \in \mathbb{N}$ be such that $-d \leq f(x) - cx \leq d$. Consider the functions

$$g_1(x) = f(x) - kx + d + 1, g_2(x) = x - f(x) + kx + d + 1.$$

Note that

$$g_1(x) \geq cx - d - kx + d + 1 = \lambda x + 1 \geq 1,$$
$$g_2(x) \geq x - (cx + d) + kx + d + 1 = (1 - \lambda)x + 1 \geq 1$$

so that these are also functions from $\mathbb{N}$ to $\mathbb{N}$. They also satisfy condition (a) for f:

$$\frac{g_1(x+a) - g_1(x)}{a} = \frac{f(x+a) - k(x+a) + d - f(x) + kx - d}{a} = \frac{f(x+a) - f(x)}{a} - k$$

is an integer if and only if $\frac{f(x+a)-f(x)}{a}$ is, which happens if and only if $a = 1$. A similar argument holds for g_2.

Now note that $g_1(x) - \lambda x = f(x) - cx + d + 1$ is bounded, and so is $g_2(x) - (1 - \lambda)x = cx - f(x) + d + 1$. So they satisfy the weaker form of condition (b) as well. Thus applying the reasoning in the second paragraph, we see that $\lambda \geq \frac{1}{2}$ and $1 - \lambda \geq \frac{1}{2}$. This forces $\lambda = \frac{1}{2}$, which finishes our proof. □

Alternative Solution. We give a different proof that $\{c\} = 1/2$. Let us first prove a claim:

Claim. For any $k \geq 1$ and any x, $f(x+2^k) - f(x)$ is divisible by 2^{k-1} but not 2^k.

Proof. We prove this via induction on k. For $k = 1$, the claim is trivial. Now assume the statement is true for some k, and note that $f(x+2^k) - f(x) = 2^{k-1}y_1$ and $f(x+2^k+2^k) - f(x+2^k) = 2^{k-1}y_2$ for some odd integers y_1, y_2. Adding these, we see that

$$f(x+2^{k+1}) - f(x) = 2^{k-1}(y_1+y_2)$$

which is divisible by 2^k because y_1+y_2 is even. The fact that this is not divisible by 2^{k+1} follows from the condition on f. □

Now using this claim, we see that for any $k \geq 1$, $f(1+2^k) = f(1)+2^{k-1}(2y_k+1)$ for some integer y_k, which means

$$f(1+2^k) - c(1+2^k) = f(1) - c + 2^k\left(y_k + \frac{1}{2} - c\right).$$

Thus $2^k(y_k + \frac{1}{2} - c)$ is bounded. But if this quantity is never zero, we have

$$2^k\left|y_k + \frac{1}{2} - c\right| = 2^{k-1}|2y_k + 1 - 2c| \geq 2^{k-1},$$

contradicting boundedness. Thus we must have $y_k + \frac{1}{2} - c = 0$ for some k, so that $\{c\} = \frac{1}{2}$. □

Problem 4. Let $k \geq 1$ and $N > 1$ be two integers. On a circle are placed $2N+1$ coins all showing heads. Calvin and Hobbes play the following game. Calvin starts and on his move can turn any coin from heads to tails. Hobbes on his move can turn at most one coin that is next to the coin that Calvin turned just now from tails to heads. Calvin wins if at any moment there are k coins showing tails after Hobbes has made his move. Determine all values of k for which Calvin wins the game.

Solution. Calvin wins if $k \in \{1, 2, \ldots, N+1\}$ and Hobbes wins otherwise.

Label the coins $1, 2, \ldots, 2N+1$. Note that if $k \geq N+2$ then Hobbes wins as follows: he pairs the coins $2i-1$ and $2i$ for $1 \leq i \leq N$. If Calvin in a move makes both coins in a pair tails, Hobbes in that move turns the one which was tails prior to Calvin's move back to heads. Thus, he can ensure that after his move, no pair has more than one tails. So, the number of tails after his move is $\leq 1 + \frac{(2N+1)-1}{2} = N+1$, hence Hobbes wins. If $k \leq N$, then Calvin wins by simply turning coins $2i$ for $1 \leq i \leq N$. Now let $k = N+1$.

Let $N = 2m+\varepsilon$ where $\varepsilon \in \{0,1\}$. Now consider m arcs on the circle with the ith arc containing $\{4i+1, 4i+2, 4i+3, 4i+4\}$, for all $0 \leq i < m$. Calvin makes $3m$ moves as follows: on move $3i+1, 3i+2$, and $3i+3$, he turns coins numbered $4i+2, 4i+4$, and $4i+3$, respectively, to tails, for all $0 \leq i < m$. Thus, no matter what Hobbes does, each arc will have either $\{4i+2, 4i+3\}$ tails (type $\widehat{23}$), or $\{4i+3, 4i+4\}$ tails (type $\widehat{34}$), or all of $\{4i+2, 4i+3, 4i+4\}$ tails (type $\widehat{234}$), by the end of these $3m$ moves. We now split into cases:

Case 1. $\varepsilon = 0$ In this case, if we have *any* arc of type $\widehat{234}$, we get that there are $\geq 3 + 2(m-1) = N+1$ tails at the end and the game is won. Assume all arcs are of type $\widehat{23}$ or $\widehat{34}$; hence we currently have $2m$ tails. Now, if $4m+1$ has no tails neighbours, Calvin turns it to win. So assume the arc $\{4m-3, 4m-2, 4m-1, 4m\}$ is of type $\widehat{34}$. If $\{1,2,3,4\}$ is also of type $\widehat{34}$, Calvin can turn 1 to tails to win as it has no tails neighbours. If it is of type $\widehat{23}$, then we must have an $0 \leq i < m-1$ such that the ith arc is of type $\widehat{23}$ but the $(i+1)$th arc is of type $\widehat{34}$, which means that Calvin can turn $4i+1$ to tails to win, as it has no tails neighbours.

Case 2. $\varepsilon = 1$ Again, if we have any arc of type $\widehat{234}$, Calvin turns $4m+2$ and we end up with $\geq 3 + 2(m-1) + 1 = N+1$ tails at the end and the game is won. Assume all arcs are of type $\widehat{23}$ or $\widehat{34}$; hence we currently have $2m$ tails. If Calvin can turn $4m+1$, then he wins, by turning $4m+3$ next move; so assume the arc $\{4m-3, 4m-2, 4m-1, 4m\}$ is of type $\widehat{34}$. If $\{1,2,3,4\}$ is of type $\widehat{34}$, then Calvin can turn 1 and $4m+2$ to secure his win; so assume

$\{1,2,3,4\}$ is of type $\widehat{23}$. Thus, there exists $0 \le i < m-1$ such that the ith arc is of type $\widehat{23}$ and the $(i+1)$th arc is of type $\widehat{34}$, hence Calvin wins by turning $4m+2$ and $4i+1$, in the next two moves.

In conclusion, Calvin wins if $k = N+1$, completing the proof. □

Problem 5. Euler marks n different points in the Euclidean plane. For each pair of marked points, Gauss writes down the number $\lfloor \log_2 d \rfloor$ where d is the distance between the two points. Prove that Gauss writes down less than $2n$ distinct values.

Note: For any $d > 0$, $\lfloor \log_2 d \rfloor$ is the unique integer k such that $2^k \le d < 2^{k+1}$.

Solution. We first prove that the Gauss writes down at most n even numbers.

For each even number $2k$ that the barista writes down, choose a single pair of points whose distance d satisfies $2^k \le d < 2^{k+1}$. Connect these points with a red edge. We claim there cannot be a cycle: indeed, if the edges corresponding to the distinct even integers $2k_1, \ldots, 2k_m, 2k_{m+1}$ form a cycle in that order, then assume without loss of generality that $2k_{m+1}$ is the largest among these, and $2k_i$ is the largest among the rest. The sum of distances for the first m edges is at most

$$2^{2k_1+1} + \cdots + 2^{2k_m+1} \le 2^{2k_i+1}(1 + 2^{-2} + 2^{-4} + \cdots) \le \frac{2^{2k_i+1}}{1-\frac{1}{2^2}} < 2^{2k_i+2} \le 2^{2k_{m+1}},$$

i.e., less than the distance corresponding to the last edge: a contradiction to triangle inequality. So there are at most $n-1$ red edges.

This implies that Gauss only writes at most $n-1$ even numbers, and similarly at most $n-1$ odd numbers. Thus, Gauss writes down at most $2n-2$ numbers in total. □

Problem 6. Euclid has a tool called *cyclos* which allows him to do the following:

- Given three non-collinear marked points, draw the circle passing through them.
- Given two marked points, draw the circle with them as endpoints of a diameter.
- Mark any intersection points of two drawn circles or mark a new point on a drawn circle.

Show that given two marked points, Euclid can draw a circle centered at one of them and passing through the other, using only the cyclos.

Solution. We begin by proving a series of lemmas.

Lemma 1. Given a non-right angled triangle ABC, we can draw the nine-point circle and mark the orthocentre H using only a cyclos.

Proof. Draw circles $(BC), (CA), (AB)$ and mark their intersections to get the three feet of altitudes D, E, F opposite A, B, C. Now draw the circle (DEF) to get the nine-point circle. Draw $(BDF), (CDE), (AEF)$ and they meet at H, which we can also mark.

Lemma 2. Given points A, B, we can mark the midpoint M of AB using only a cyclos.

Proof. Draw the circle (AB) and choose a point X on it. Draw circles $(XA), (XB)$ and mark their intersection Y. Now mark a point Z on the circle (XA) apart from the marked points. Clearly, Z does not lie on AB nor on (AB), hence we can draw (AZB). Mark five points $Z_1, \ldots, Z_5$ on this circle, each different from all previous points and verify if either A lies on (Z_iB) or B lies on (Z_iA) for each $1 \le i \le 5$ before marking the new point. By pigeonhole principle, for some three indices i, j, k, the three triangles AZ_iB are non-right angled, hence we can draw their ninepoint circles by Lemma 1. All of them pass through M, and their centres are not collinear, else homothety at the centre of (ABZ) implies the orthocentres of the three triangles are collinear; but they all lie on the reflection of (AZB) in AB, a contradiction! Thus, these three nine-point circles meet at only M, and we mark this point.

Lemma 3. Given points A, B, C, D on the plane in general position, we can mark the intersection point E of lines AB and CD using only a cyclos.

Proof. Draw (AB) and mark five points on it, all different from previously marked points. For each marked point X; draw (CX) and (DX) and check whether they have an intersection apart from X (i.e., if they are tangent, or if X lies on CD). We can find three points X_1, X_2, X_3 among them not lying on CD. Denote by Y_i the second intersection of $(AX_i), (BX_i)$ and by Z_i the second intersection of $(CX_i), (DX_i)$ and mark them, for each $1 \leq i \leq 3$. Draw the circles $(X_iY_iZ_i)$ and note that they all pass through E and have diameters EX_i for all i; so they are not coaxial as X_1, X_2, X_3 are not collinear; all lying on (AB). Thus we mark E as the unique point common to them all. (Note: if C, D lie on (AB), we can pick a point T on it other than these four, then a point X on (AT), and continue the same argument again, avoiding all edge cases.)

Lemma 4. Given a circle Γ, we can mark the centre of Γ using only a cyclos.

Proof. Mark points $A, B, C \in \Gamma$ and mark the midpoints of BC, CA, AB to get A_1, B_1, C_1 according to Lemma 2. Draw the circles $(AB_1C_1), (BA_1C_1), (CB_1A_1)$ and mark the intersection to get the centre of Γ.

Lemma 5. Given a circle Γ and point A on Γ, we can mark a point K such that line AK is tangent to Γ using only a cyclos.

Proof. Mark points B_1, B_2, B_3 and C on Γ. By Lemma 4, mark the point O, the centre of Γ. Draw (B_iO) and (AOC) and mark the intersection denoted F_i; there exists an index j for which $F_j \neq O$; mark the point K which is the intersection of B_jF_j and OM where M is the midpoint of AC (which we mark by Lemma 2) by Lemma 3. Clearly, K lies on A tangent to Γ. Note that we can do this again to get multiple such points K by choosing different C each time.

Lemma 6. Given a circle Γ and a point A on Γ, and a point B not on Γ, we can mark the point C which is the second intersection of line AB and Γ using only a cyclos.

Proof. Mark the foot of perpendicular M from O onto line AB as done in Lemma 1. Mark the intersection of line OM and AK by Lemma 3, where K is a point on the A-tangent to Γ as constructed in Lemma 5. Draw (OAK) and mark the second intersection with Γ to obtain C.

Lemma 7. Given points A, B, C not all on a line, we can draw the reflection of A in BC using only a cyclos.

Proof. Draw (ABC) and mark the orthocentre H of ABC by Lemma 1. Mark the intersection A' of line AH with (BHC) using Lemma 6, which is the A-reflection in line BC.

Lemma 8. Given points A, B, we can mark the point C which is the reflection of A in B using only a cyclos.

Proof. Draw (AB), and by Lemma 6, mark two points K_1, K_2 such that BK_i is tangent to (AB) for $i \in \{1, 2\}$. By Lemma 7, mark the reflection C of A in line K_1K_2 as desired.

Thus, a cyclos can do everything a compass can: to draw a circle with given centre A and given radius B, we use Lemma 8 to mark the reflection C of B in A and use the cyclos to draw (BC) which has centre A and passes through B. □

Alternative solution. After deriving the first two lemmas in the previous solution, one can proceed as follows: Let C denote the point such that A is the midpoint of CB.

Choose a generic point X. We can get the midpoint M of BX and the foot of perpendicular D from X to AB. Draw the circle passing through A, D and M. This is the nine-point of circle of triangle CBX. Intersect this circle with the circle whose diameter is BX. The intersection point other than D is the foot of perpendicular E from B to CX. Note that $|AE| = |AB|$. Similarly, we can construct another point F such that $|AF| = |AB|$. The circle through B, E and F is the required circle. $\square$