Solutions

1. Let $ABC$ be a triangle with integer sides in which $AB < AC$. Let the tangent to the circumcircle of triangle $ABC$ at $A$ intersect the line $BC$ at $D$. Suppose $AD$ is also an integer. Prove that $\gcd(AB, AC) > 1$.

**Solution:** We may assume that $B$ lies between $C$ and $D$. Let $AB = c$, $BC = a$ and $CA = b$. Then $b > c$. Let $BD = x$ and $AD = y$. Observe that $\angle DAB = \angle DCA$. Hence $\triangle DAB \sim \triangle DCA$. We get

$$\frac{x}{y} = \frac{c}{b} = \frac{y}{x + a}.$$ 

Therefore $xb = yc$ and $by = c(x + a)$. Eliminating $x$, we get $y = abc / (b^2 - c^2)$.

Suppose $\gcd(b, c) = 1$. Then $\gcd(b, b^2 - c^2) = 1 = \gcd(c, b^2 - c^2)$. Since $y$ is an integer, $b^2 - c^2$ divides $a$. Therefore $b + c$ divides $a$. Hence

$$a \geq b + c.$$ 

This contradicts triangle inequality. We conclude that $\gcd(b, c) > 1$.

2. Let $n$ be a natural number. Find all real numbers $x$ satisfying the equation

$$\sum_{k=1}^{n} \frac{kx^k}{1 + x^{2k}} = \frac{n(n + 1)}{4}.$$ 

**Solution:** Observe that $x \neq 0$. We also have

$$\frac{n(n + 1)}{4} = \left| \sum_{k=1}^{n} \frac{kx^k}{1 + x^{2k}} \right| \leq \sum_{k=1}^{n} \frac{k|x|^k}{1 + x^{2k}}.$$ 

$$= \sum_{k=1}^{n} \frac{k}{1 + |x|^k}.$$ 

$$\leq \sum_{k=1}^{n} \frac{k}{2} = \frac{n(n + 1)}{4}.$$ 

Hence equality holds every where. It follows that $x = |x|$ and $|x| = 1/|x|$. We conclude that $x = 1$ is the unique solution to the equation.

3. For a rational number $r$, its *period* is the length of the smallest repeating block in its decimal expansion. For example, the number $r = 0.123123123\ldots$ has period 3. If $S$ denotes the set of all rational numbers $r$ of the form $r = 0.abcdefghfgh$ having period 8, find the sum of all the elements of $S$.

**Solution:** Let us first count the number of elements in $S$. There are $10^8$ ways of choosing a block of length 8. Of these, we should not count the blocks of the form $abcdabcd$, $abababab$, and $aaaaaaaa$. There are $10^4$ blocks of the form $abcdabcd$. They include blocks of the form $abababab$ and $aaaaaaaa$. Hence the blocks of length exactly 8 is $10^8 - 10^4 = 99990000$. 
For each block $abcdefg$ consider the block $a'b'c'd'e'f'g'h'$ where $x' = 9 - x$. Observe that whenever $0.abcdefg$ is in $S$, the rational number $0.a'b'c'd'e'f'g'h'$ is also in $S$. Thus every element $0.abcdefg$ of $S$ can be uniquely paired with a distinct element $0.a'b'c'd'e'f'g'h'$ of $S$. We also observe that

\[
0.abcdefg + 0.a'b'c'd'e'f'g'h' = 0.99999999 = 1.
\]

Hence the sum of elements in $S$ is

\[
\frac{99990000}{2} = 49995000.
\]

4. Let $E$ denote the set of 25 points $(m, n)$ in the xy-plane, where $m, n$ are natural numbers, $1 \leq m \leq 5$, $1 \leq n \leq 5$. Suppose the points of $E$ are arbitrarily coloured using two colours, red and blue. Show that there always exist four points in the set $E$ of the form $(a, b), (a + k, b), (a + k, b + k), (a, b + k)$ for some positive integer $k$ such that at least three of these four points have the same colour. (That is, there always exist four points in the set $E$ which form the vertices of a square and having at least three points of the same colour.)

**Solution:** Name the points from bottom row to top (and from left to right) as $A_j, B_j, C_j, D_j, E_j, 1 \leq j \leq 5$.

Note that among 5 points $A_1, B_1, C_1, D_1, E_1$, there are at least 3 points of the same colour, say, red. (This follows from pigeonhole principle.) We consider several cases: (the argument holds irrespective of the colour assigned to the other two points.)

(I) Take three adjacent points having the same colour. (e.g. $A_1, B_1, C_1$ or $B_1, C_1, D_1$.) The argument is similar in both the cases. If $A_1, B_1, C_1$ are red then $A_2, B_2, C_2$ are all blue; otherwise we get a square having three red vertices. The same reasoning shows that $A_3, B_3, C_3$ are all red. Now $A_1, C_1, A_3, C_3$ have all red vertices.

(II) Three alternate points $A_1, C_1, E_1$ which are red: Then $A_3, C_3, E_3$ have to be blue; otherwise, we get a square with three red vertices. Same reasoning shows that $A_5, C_5, E_5$ are red. Therefore we have $A_1, E_1, A_5, E_5$ have red colour.

(III) Only two adjacent points having red colour: There are three sub cases.
(a) $A_1, B_1, D_1$ red: In this case $A_2, B_2$ are blue and therefore $A_3, B_3$ are red. But then $B_1, D_1, B_3$ are red vertices of a square.
(b) $B_1, C_1, E_1$ are red. This is similar to case (a).
(c) $A_1, B_1, E_1$ are red. We successively have $A_2, B_2$ blue; $A_3, B_3$ red; $A_4, B_4$ blue; $A_5, B_5$ red. Now $A_1, E_1, A_3$ are the red vertices of a square.
These are the only essential cases and all other reduce to one of these cases.

5. Find all natural numbers $n$ such that $1 + \lceil \sqrt{2n} \rceil$ divides $2n$. (For any real number $x$, $\lceil x \rceil$ denotes the largest integer not exceeding $x$.)

**Solution:** Let $\lceil \sqrt{2n} \rceil = k$. We observe that $x - 1 < \lceil x \rceil \leq x$. Hence

\[
\sqrt{2n} < 1 + k \leq 1 + \sqrt{2n}.
\]
Divisibility gives \((1 + k)d = 2n\) for some positive integer \(d\). Therefore we obtain
\[
\sqrt{2n} < \frac{2n}{d} \leq 1 + \sqrt{2n}.
\]
The first inequality gives \(d < \sqrt{2n} < 1 + k\). But then
\[
d = \frac{2n}{1 + k} = \frac{(\sqrt{2n})^2}{1 + k} \geq \frac{k^2}{1 + k} = (k - 1) + \frac{1}{k + 1} > k - 1.
\]
We thus obtain \(k - 1 < d < k + 1\). Since \(d\) is an integer, it follows that \(d = k\). This implies that \(n = k(k + 1)/2\). Thus \(n\) is a triangular number. It is easy to check that every triangular number is a solution.

6. Let \(ABC\) be an acute-angled triangle with \(AB < AC\). Let \(I\) be the incentre of triangle \(ABC\), and let \(D, E, F\) be the points at which its incircle touches the sides \(BC, CA, AB\), respectively. Let \(BI, CI\) meet the line \(EF\) at \(Y, X\), respectively. Further assume that both \(X\) and \(Y\) are outside the triangle \(ABC\). Prove that
(i) \(B, C, Y, X\) are concyclic; and
(ii) \(I\) is also the incentre of triangle \(DYX\).

**Solution:**

(a) We first show that \(BIFX\) is a cyclic quadrilateral. Since \(\angle BIC = 90^\circ + (A/2)\), we see that \(\angle BIX = 90^\circ - (A/2)\). On the otherhand \(FAE\) is an isosceles triangle so that \(\angle AFE = 90^\circ - (A/2)\). But \(\angle AFE = \angle BFX\) as they are vertically opposite angles. Therefore \(\angle BFX = 90^\circ - (A/2) = \angle BIX\). It follows that \(BIFX\) is a cyclic quadrilateral. Therefore \(\angle BXI = \angle BFI\). But \(\angle BFI = 90^\circ\) since \(IF \perp AB\). We obtain \(\angle BXC = \angle BXI = 90^\circ\).

A similar consideration shows that \(\angle BYC = 90^\circ\). Therefore \(\angle BXC = \angle BYC\) which implies that \(BCYX\) is a cyclic quadrilateral.

(b) We also observe that \(BDIX\) is a cyclic quadrilateral as \(\angle BXI = 90^\circ = \angle BDI\) and therefore \(\angle BXI + \angle BDI = 180^\circ\). This gives \(\angle DXI = \angle DBI = B/2\). Now the concyclic-ity of \(B, I, F, X\) shows that \(\angle IXF = \angle IBF = B/2\). Hence \(\angle DXI = \angle IXF\). Hence \(XI\) bisects \(\angle DXY\). Similarly, we can show that \(YI\) bisects \(\angle DYX\). It follows that \(I\) is the incentre of \(\triangle DYX\) as well.