Regional Mathematical Olympiad-2017

Solutions

1. Let $AOB$ be a given angle less than $180^\circ$ and let $P$ be an interior point of the angular region determined by $\angle AOB$. Show, with proof, how to construct, using only ruler and compasses, a line segment $CD$ passing through $P$ such that $C$ lies on the ray $OA$ and $D$ lies on the ray $OB$, and $CP : PD = 1 : 2$.

**Solution:** Draw a line parallel to $OA$ through $P$. Let it intersect $OB$ in $M$. Using compasses, draw an arc of a circle with centre $M$ and radius $MO$ to cut $OB$ in $L$, $L \neq O$. Again with $L$ as centre and with the same radius $OM$ draw one more arc of a circle to cut $OB$ in $D$, $D \neq M$. Join $DP$ and extend it to meet $OA$ in $C$. Then $CD$ is the required line segment such that $CP : PD = 1 : 2$. This follows from similar triangles $OCD$ and $MPD$.

2. Show that the equation
\[ a^3 + (a + 1)^3 + (a + 2)^3 + (a + 3)^3 + (a + 4)^3 + (a + 5)^3 + (a + 6)^3 = b^4 + (b + 1)^4 \]
has no solutions in integers $a, b$.

**Solution:** We use divisibility argument by $7$. Observe that the remainders of seven consecutive cubes modulo $7$ are $0, 1, 1, 6, 1, 6, 6$ in some (cyclic) order. Hence the sum of seven consecutive cubes is $0$ modulo $7$. On the other hand the remainders of two consecutive fourth powers modulo $7$ is one of the sets $\{0,1\}, \{1,2\}, \{2,4\}, \{4,4\}$. Hence the sum of two fourth powers is never divisible by $7$. It follows that the given equation has no solution in integers.
3. Let \( P(x) = x^2 + \frac{1}{2}x + b \) and \( Q(x) = x^2 + cx + d \) be two polynomials with real coefficients such that \( P(x)Q(x) = Q(P(x)) \) for all real \( x \). Find all the real roots of \( P(Q(x)) = 0 \).

**Solution:** Observe that

\[
P(x)Q(x) = x^4 + \left( c + \frac{1}{2} \right)x^3 + \left( b + \frac{c}{2} + d \right)x^2 + \left( \frac{d}{2} + bc \right)x + bd.
\]

Similarly,

\[
Q(P(x)) = \left( x^2 + \frac{1}{2}x + b \right)^2 + c \left( x^2 + \frac{1}{2}x + b \right) + d
\]

\[
= x^4 + x^3 + \left( 2b + \frac{1}{4} + c \right)x^2 + \left( b + \frac{c}{2} \right)x + b^2 + bc + d.
\]

Equating coefficients of corresponding powers of \( x \), we obtain

\[
c + \frac{1}{2} = 1, \quad b + \frac{c}{2} + d = 2b + \frac{1}{4} + c, \quad \frac{d}{2} + bc = b + \frac{c}{2}, \quad b^2 + bc + d = bd.
\]

Solving these, we obtain

\[
c = \frac{1}{2}, \quad d = 0, \quad b = -\frac{1}{2}.
\]

Thus the polynomials are

\[
P(x) = x^2 + \frac{1}{2}x - \frac{1}{2}, \quad Q(x) = x^2 + \frac{1}{2}x.
\]

Therefore,

\[
P(Q(x)) = \left( x^2 + \frac{1}{2}x \right)^2 + \frac{1}{2} \left( x^2 + \frac{1}{2}x \right) - \frac{1}{2}
\]

\[
= x^4 + x^3 + \frac{3}{4}x^2 + \frac{1}{4}x - \frac{1}{2}.
\]

It is easy to see that

\[
P(Q(-1)) = 0, \quad P(Q(1/2)) = 0.
\]

Thus \((x + 1)\) and \((x - 1/2)\) are factors of \( P(Q(x)) \). The remaining factor is

\[
h(x) = x^2 + \frac{1}{2}x + 1.
\]

The discriminant of \( h(x) \) is \( D = (1/4) - 4 < 0 \). Hence \( h(x) = 0 \) has no real roots. Therefore the only real roots of \( P(Q(x)) = 0 \) are \(-1\) and \(1/2\).
4. Consider \( n^2 \) unit squares in the \( xy \)-plane centred at point \((i, j)\) with integer coordinates, \(1 \leq i \leq n, 1 \leq j \leq n\). It is required to colour each unit square in such a way that whenever \(1 \leq i < j \leq n\) and \(1 \leq k < l \leq n\), the three squares with centres at \((i, k), (j, k), (j, l)\) have distinct colours. What is the least possible number of colours needed?

**Solution:** We first show that at least \(2n - 1\) colours are needed. Observe that squares with centres \((i, 1)\) must all have different colours for \(1 \leq i \leq n\); let us call them \(C_1, C_2, \ldots, C_n\). Besides, the squares with centres \((n, j)\), for \(2 \leq j \leq n\) must have different colours and these must be different from \(C_1, C_2, \ldots, C_n\). Thus we need at least \(n + (n - 1) = 2n - 1\) colours. The following diagram shows that \(2n - 1\) colours will suffice.

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>n</th>
<th>n + 1</th>
<th>n + 2</th>
<th>2n - 2</th>
<th>2n - 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>n + 1</td>
<td>n + 2</td>
<td>2n - 2</td>
<td>2n - 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>n - 1</td>
<td>n</td>
<td>n + 1</td>
<td>2n - 2</td>
<td>2n - 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5. Let \( \Omega \) be a circle with a chord \(AB\) which is not a diameter. Let \( \Gamma_1 \) be a circle on one side of \(AB\) such that it is tangent to \(AB\) at \(C\) and internally tangent to \(\Omega\) at \(D\). Likewise, let \( \Gamma_2 \) be a circle on the other side of \(AB\) such that it is tangent to \(AB\) at \(E\) and internally tangent to \(\Omega\) at \(F\). Suppose the line \(DC\) intersects \(\Omega\) at \(X \neq D\) and the line \(FE\) intersects \(\Omega\) at \(Y \neq F\). Prove that \(XY\) is a diameter of \(\Omega\).

**Solution:** Draw the tangent \(PQ\) at \(D\) such that \(D\) is between \(P\) and \(Q\). Join \(D\) to \(A, B\) and \(C\). Let \(L = DA \cap \Gamma_1\) and \(M = DB \cap \Gamma_1\). Join \(C\) to \(L\) and \(M\). Observe that

\[
\angle ADP = \angle LMD = \angle ABD. \tag{1}
\]

Therefore \(LM\) is parallel to \(AB\) and hence \(\angle LMC = \angle MCB\) (alternate angles). Again observe that

\[
\angle ADC = \angle LDC = \angle LMC = \angle MCB = \angle MDC = \angle BDC. \tag{2}
\]
Thus $CD$ bisects $\angle ADB$. Hence $X$ is the midpoint of the arc $AB$ not containing $D$. Similarly $Y$ is the midpoint of the arc $AB$ not containing $F$. Thus the arc $XY$ is half of the sum of two arcs that together constitute the circumference of $\Omega$ and hence it is a diameter.

6. Let $x, y, z$ be real numbers, each greater than 1. Prove that

$$\frac{x+1}{y+1} + \frac{y+1}{z+1} + \frac{z+1}{x+1} \leq \frac{x-1}{y-1} + \frac{y-1}{z-1} + \frac{z-1}{x-1}.$$  

**Solution:** We may assume that $x = \max\{x, y, z\}$. There are two cases: $x \geq y \geq z$ and $x \geq z \geq y$. We consider both these cases. The inequality is equivalent to

$$\left\{ \frac{x-1}{y-1} - \frac{x+1}{y+1} \right\} + \left\{ \frac{y-1}{z-1} - \frac{y+1}{z+1} \right\} + \left\{ \frac{z-1}{x-1} - \frac{z+1}{x+1} \right\} \geq 0.$$  

This is further equivalent to

$$\frac{x-y}{y^2-1} + \frac{y-z}{z^2-1} + \frac{z-x}{x^2-1} \geq 0.$$  

Suppose $x \geq y \geq z$. We write

$$\frac{x-y}{y^2-1} + \frac{y-z}{z^2-1} + \frac{z-x}{x^2-1} = \frac{x-y}{y^2-1} + \frac{y-z}{z^2-1} + \frac{z-y}{x-1} - \frac{x-1}{x^2-1}.$$  

This reduces to

$$\frac{(x-y)(x^2-y^2)}{(x^2-1)(y^2-1)} + \frac{(y-z)(x^2-z^2)}{(x^2-1)(z^2-1)}.$$  

4
Since $x \geq y$ and $x \geq z$, this sum is nonnegative.

Suppose $x \geq z \geq y$. We write
\[
\frac{x - y}{y^2 - 1} + \frac{y - z}{z^2 - 1} + \frac{z - x}{x^2 - 1} = \frac{x - z + z - y}{y^2 - 1} + \frac{y - z}{z^2 - 1} + \frac{z - x}{x^2 - 1}.
\]
This reduces to
\[
(x - z) \frac{(x^2 - y^2)}{(x^2 - 1)(y^2 - 1)} + (z - y) \frac{(z^2 - y^2)}{(y^2 - 1)(z^2 - 1)}.
\]
Since $x \geq z$ and $z \geq y$, this sum is nonnegative.

Thus
\[
\frac{x - y}{y^2 - 1} + \frac{y - z}{z^2 - 1} + \frac{z - x}{x^2 - 1} \geq 0
\]
in both the cases. This completes the proof.