1. In the given figure, $ABCD$ is a square paper. It is folded along $EF$ such that $A$ goes to a point $A' \neq C, B$ on the side $BC$ and $D$ goes to $D'$. The line $A'D'$ cuts $CD$ in $G$. Show that the inradius of the triangle $GCA'$ is the sum of the inradii of the triangles $GD'F$ and $A'BE$.

**Solution:** Observe that the triangles $GCA'$ and $A'BE$ are similar to the triangle $GD'F$. If $GF = u, GD' = v$ and $D'F = w$, then we have

$$A'G = pu, CG = pv, A'C = pw, A'E = qu, BE = qw, A'B = qv.$$  

If $r$ is the inradius of $\triangle GD'F$, then $pr$ and $qr$ are respectively the inradii of triangles $GCA'$ and $A'BE$. We have to show that $pr = r + qr$. We also observe that $AE = EA'$, $DF = FD'$.

Therefore

$$pw + qv = qw + qu = w + u + pv = v + pu.$$  

The last two equalities give $(p-1)(u-v) = w$. The first two equalities give $(p-q)w = q(u-v)$. Hence

$$\frac{p-q}{q} = \frac{u-v}{w} = \frac{1}{p-1}.$$  

This simplifies to $p(p-q-1) = 0$. Since $p \neq 0$, we get $p = q+1$. This implies that $pr = qr + r$.

2. Suppose $n \geq 0$ is an integer and all the roots of $x^3 + \alpha x + 4 - (2 \times 2016^n) = 0$ are integers. Find all possible values of $\alpha$.

**Solution 1:** Let $u, v, w$ be the roots of $x^3 + \alpha x + 4 - (2 \times 2016^n) = 0$. Then $u + v + w = 0$ and $uvw = -4 + (2 \times 2016^n)$. Therefore we obtain

$$uvw(u + v) = 4 - (2 \times 2016^n).$$  

Suppose $n \geq 1$. Then we see that $uvw(u + v) \equiv 4 \pmod{2016^n}$. Therefore $uvw(u + v) \equiv 1 \pmod{3}$ and $uvw(u + v) \equiv 1 \pmod{9}$. This implies that $u \equiv 2 \pmod{3}$ and $v \equiv 2 \pmod{3}$. This shows that modulo 9 the pair $(u, v)$ could be any one of the following:

$$(2, 2), (2, 5), (2, 8), (5, 2), (5, 5), (5, 8), (8, 2), (8, 5), (8, 8).$$  

In each case it is easy to check that $uvw(u + v) \not\equiv 4 \pmod{9}$. Hence $n = 0$ and $uvw(u + v) = 2$. It follows that $(u, v) = (1, 1), (1, -2)$ or $(-2, 1)$. Thus

$$\alpha = uv + vw + wu = uv - (u + v)^2 = -3$$  

for every pair $(u, v)$. 


**Solution 2:** Let \( a, b, c \in \mathbb{Z} \) be the roots of the given equation for some \( n \in \mathbb{N}_0 \). By Vieta Theorem, we know that

\[
\begin{align*}
  a + b + c &= 0 \\
  ab + bc + ca &= \alpha \\
  abc &= 2 \times 2016^n - 4
\end{align*}
\]

If possible, let us have \( n \geq 1 \). Since \( 7|2016 \), we have that

\[
7|abc + 4 \implies 7|3(abc + 4) \implies 7|3abc + 12 \implies 7|3abc + 5
\]

Since we have \( a + b + c = 0 \), we get that \( 3abc = a^3 + b^3 + c^3 \). Substituting this in the earlier expression, we get that

\[
a^3 + b^3 + c^3 + 5 \equiv 0 \pmod{7}
\]

Consider below, a table calculating the residues of cubes modulo 7.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^3 )</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
</tbody>
</table>

Hence, we know that if \( x \in \mathbb{N} \), then we have \( x^3 \equiv 0, 1, -1 \pmod{7} \). Since \( a^3 + b^3 + c^3 \equiv 2 \pmod{7} \), we see that we must have one of the numbers divisible by 7 and the other two numbers, when cubed, must leave 1 as remainder modulo 7. Without of generality, let us assume that

\[
a \equiv 0 \pmod{7}, \quad b^3, c^3 \equiv 1 \pmod{7}
\]

Hence, we have \( b, c \equiv 1, 2, 4 \pmod{7} \). We will consider all possible values of \( b + c \) modulo 7. Since the expression is symmetric in \( b, c \), modulo 7, we will consider \( b \leq c \).

<table>
<thead>
<tr>
<th>( b )</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>2</th>
<th>2</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c )</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>2</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>( b + c )</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>4</td>
<td>6</td>
<td>1</td>
</tr>
</tbody>
</table>

We see that, in all the above cases, we get \( 7|b + c \). But this is a contradiction, since \( 7|a+b+c \) and \( 7|a \) together imply that \( 7|b + c \). Hence, we cannot have \( n \geq 1 \). Hence, the only possible value is \( n = 0 \). Substituting this value in the original equation, the equation becomes

\[
x^3 + \alpha x + 2 = 0
\]

Solving the equations \( a + b + c = 0 \) and \( abc = -2 \) in integers, we see that the only possible solutions \( (a, b, c) \) are permutations of \((1, 1, -2)\). In case of any permutation, \( \alpha = -3 \). Substituting this value of \( \alpha \) back in the equation, we see that we indeed, get integer roots. Hence, the only possible value for \( \alpha \) is \(-3\).

3. Find the number of triples \((x, a, b)\) where \( x \) is a real number and \( a, b \) belong to the set \( \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \) such that

\[
x^2 - a\{x\} + b = 0,
\]

where \( \{x\} \) denotes the fractional part of the real number \( x \). (For example \( \{1.1\} = 0.1 = \{-0.9\} \).)
Solution: Let us write \( x = n + f \) where \( n = [x] \) and \( f = \{x\} \). Then

\[
f^2 + (2n - a)f + n^2 + b = 0. \tag{1}
\]

Observe that the product of the roots of (1) is \( n^2 + b \geq 1 \). If this equation has to have a solution \( 0 \leq f < 1 \), the larger root of (1) is greater 1. We conclude that the equation (1) has a real root less than 1 only if \( P(1) < 0 \) where \( P(y) = y^2 + (2n - a)y + n^2 + 2b \). This gives

\[
1 + 2n - a + n^2 + 2b < 0.
\]

Therefore we have \((n + 1)^2 + b < a\). If \( n \geq 2 \), then \((n + 1)^2 + b \geq 10 > a\). Hence \( n \leq 1 \). If \( n = -4 \), then again \((n + 1)^2 + b \geq 10 > a\). Thus we have the range for \( n: -3, -2, -1, 0, 1 \).

If \( n = -3 \) or \( n = 1 \), we have \((n + 1)^2 = 4\). Thus we must have \( 1 + 2n - a + n^2 + 2b < 0 \).

Therefore we have \((n + 1)^2 + b < a\). If \( n = -2 \) and \( n = 0 \), we have \((n + 1)^2 = 1\). Hence we require \( 1 + b < a \). We again count pairs \((a, b)\) such that \( a - b > 1 \). For \( a = 9 \), we get 7 values of \( b \); for \( a = 8 \) we get 6 values of \( b \) and so on. Thus we get \( 2(7 + 6 + 5 + 4 + 3 + 2 + 1) = 56 \) values for the triple \((x, a, b)\).

Thus the total number of triples \((x, a, b)\) is \( 112 \).

4. Let \( ABCDE \) be a convex pentagon in which \( \angle A = \angle B = \angle C = \angle D = 120^\circ \) and whose side lengths are 5 consecutive integers in some order. Find all possible values of \( AB + BC + CD \).

Solution 1: Let \( AB = a, BC = b, \) and \( CD = c \). By symmetry, we may assume that \( c < a \). We show that \( DE = a + b \) and \( EA = b + c \).

Draw a line parallel to \( BC \) through \( D \). Extend \( EA \) to meet this line at \( F \). Draw a line parallel to \( CD \) through \( B \) and let it intersect \( DF \) in \( G \). Let \( AB \) intersect \( DF \) in \( H \). We have \( \angle FDE = 60^\circ \) and \( \angle E = 60^\circ \). Hence \( EFD \) is an equilateral triangle. Similarly \( AFH \) and \( BGF \) are also equilateral triangles. Hence \( HG = GB = c \). Moreover, \( DG = b \). Therefore \( HD = b + c \). But \( HD = AE \) since \( FH = FA \) and \( FD = FE \). Also \( AH = a - BH = a - BG = a - c \). Hence \( ED = EF = EA + AF = b + c + AH = (b + c) + (a - c) = b + a \).

We have five possibilities:

(1) \( b < c < a < b + c < a + b \);
5. Let $ABC$ be three consecutive integers provided $b = 2$. Hence we get $c = 3$ and $a = 4$. In this case $b + c = 5$ and $a + b = 6$ so that we have five consecutive integers $2, 3, 4, 5, 6$ as side lengths. In (2), $b < a < b + c$ form three consecutive integers only when $c = 2$. Hence $b = 3$, $a = 4$. But then $b + c = 5$ and $a + b = 7$. Thus the side lengths are $2, 3, 4, 6, 7$ which are not consecutive integers. In case (3), $b < b + c$ are two consecutive integers so that $c = 1$. Hence $a = 2$ and $b = 3$. We get $b + c = 4$ and $a + b = 5$ so that the consecutive integers $1, 2, 3, 4, 5$ form the side lengths. In case (4), we have $c < b + c$ as two consecutive integers and hence $b = 1$. Therefore $c = 2$, $b + c = 3$, $a = 4$ and $a + b = 5$ which is admissible. Finally, in case (5) we have $b < b + c$ as two consecutive integers, so that $c = 1$. Thus $b = 2$, $b + c = 3$, $a = 4$ and $a + b = 6$. We do not get consecutive integers.

Therefore the only possibilities are $(a, b, c) = (4, 2, 3), (2, 3, 1)$ and $(4, 1, 2)$. This shows that $a + b + c = 9, 6$ or 7. Thus there are three possible sums $AB + BC + CA$, namely, 6, 7 or 9.

Solution 2: As in the earlier solution, $ED = d = a + b$ and $EA = e = b + c$. Let the sides be $x - 2, x - 1, x, x + 1, x + 2$. Then $x \geq 3$. We also have $x + 2 \geq x - 1 + x - 2$ so that $x \leq 5$. Thus $x = 3, 4$ or 5. If $x = 5$, the sides are $\{3, 4, 5, 6, 7\}$ and here we do not have two pairs which add to a number in the set. Hence $x = 3$ or 4 and we get the sets as $\{1, 2, 3, 4, 5\}$ or $\{2, 3, 4, 5, 6\}$. With the set $\{1, 2, 3, 4, 5\}$ we get

$$(a, b, c, d, e) = (2, 3, 1, 5, 4), (4, 1, 2, 5, 3).$$

From the set $\{2, 3, 4, 5, 6\}$, we get $(a, b, c, d, e) = (4, 2, 3, 6, 5)$. Thus we see that $a + b + c = 6, 7$ or 9.

Solution 3: We use the same notations and we get $d = a + b$ and $e = b + c$. If $a \geq 5$, we see that $d - b \geq 5$. But the maximum difference in a set of 5 consecutive integers is 4. Hence $a \leq 4$. Similarly, we see $b \leq 4$ and $c \leq 4$. Thus we see that $a + b + c \leq 2 + 3 + 4 = 9$. But $a + b + c \geq 1 + 2 + 3 = 6$. It follows that $a + b + c = 6, 7, 8$ or 9. If we take $(a, b, c, d, e) = (1, 3, 2, 4, 5)$, we get $a + b + c = 6$. Similarly, $(a, b, c, d, e) = (2, 1, 4, 3, 5)$ gives $a + b + c = 7$. For $a + b + c = 8$, the only we we can get $1 + 3 + 4 = 8$. Here we cannot accommodate 2 and consecutiveness is lost. For 9, we can have $(a, b, c, d, e) = (3, 2, 4, 5, 6)$ and $a + b + c = 9$.

5. Let $ABC$ be a triangle with $\angle A = 90^\circ$ and $AB < AC$. Let $AD$ be the altitude from $A$ on to $BC$. Let $P, Q$ and $I$ denote respectively the incentres of triangles $ABD$, $ACD$ and $ABC$. Prove that $AI$ is perpendicular to $PQ$ and $AI = PQ$.

Solution: Draw $PS \parallel BC$ and $QS \parallel AD$. Then $PSQ$ is a right-angled triangle with $\angle PSQ = 90^\circ$. Observe that $PS = r_1 + r_2$ and $SQ = r_2 - r_1$, where $r_1$ and $r_2$ are the inradii of triangles $ABD$ and $ACD$, respectively. We observe that triangles $DAB$ and $DCA$ are similar to triangle $ACB$.
Hence
\[ r_1 = \frac{c}{a}, \quad r_2 = \frac{b}{a}, \]
where \( r \) is the inradius of triangle \( ABC \). Thus we get
\[ \frac{PS}{SQ} = \frac{r_2 + r_1}{r_2 - r_1} = \frac{b + c}{b - c}. \]
On the other hand \( AD = h = \frac{bc}{a} \). We also have \( BE = \frac{ca}{b + c} \) and
\[ BD^2 = c^2 - h^2 = c^2 - \frac{b^2c^2}{a^2} = \frac{c^4}{a^2}. \]
Hence \( BD = \frac{c^2}{a} \). Therefore
\[ DE = BE - BD = \frac{cb(b - c)}{a(b + c)}. \]
Thus we get
\[ \frac{AD}{DE} = \frac{b + c}{b - c} = \frac{PS}{SQ}. \]
Since \( \angle ADE = 90^\circ = \angle PSQ \), we conclude that \( \triangle ADE \sim \triangle PSQ \). Since \( AD \perp PS \), it follows that \( AE \perp PQ \).
We also observe that
\[ PQ^2 = PS^2 + SQ^2 = (r_2 + r_1)^2 + (r_2 - r_1)^2 = 2(r_1^2 + r_2^2). \]
However
\[ r_1^2 + r_2^2 = \frac{c^2 + b^2}{a^2} - r^2 = r^2. \]
Hence \( PQ = \sqrt{2}r \). We also observe that \( AI = \frac{r}{\sin(A/2)} = \frac{r}{\sin(45^\circ)} = \sqrt{2}r \). Thus \( PQ = AI \).

**Solution 2:** In the figure, we have made the construction as mentioned in the hint. Since \( P, Q \) are the incentres of \( \triangle ABD, \triangle ACD, DP, DQ \) are the internal angle bisectors of \( \angle ADB, \angle ADC \) respectively. Since \( AD \) is the altitude on the hypotenuse \( BC \) in \( \triangle ABC \), we have that \( \angle PDQ = 45^\circ + 45^\circ = 90^\circ \). It also implies that
\[ \triangle ABC \sim \triangle DBA \sim \triangle DAC \]
This implies that all corresponding length in the above mentioned triangles have the same ratio.
In particular,
\[
\frac{AI}{BC} = \frac{DP}{AB} = \frac{DQ}{AC}
\]
\[
\Rightarrow \frac{AI^2}{BC^2} = \frac{DP^2}{AB^2} = \frac{DQ^2}{AC^2} = \frac{DP^2 + DQ^2}{AB^2 + AC^2}
\]
\[
\Rightarrow \frac{AI^2}{BC^2} = \frac{PQ^2}{BC^2}, \text{ by Pythagoras Theorem in } \triangle ABC, \triangle PDQ
\]
\[
\Rightarrow AI = PQ
\]
as required.

For the second part, we note that from the above relations, we have \(\triangle ABC \sim \triangle PDQ\). Let us take \(\angle ACB = \theta\). Then, we get
\[
\angle PSD = 180^\circ - (\angle SPD + \angle SDP) = 180^\circ - (90^\circ - \theta + 45^\circ) = 45^\circ + \theta
\]
This gives us that
\[
\angle ARS = 180^\circ - (\angle ASR + \angle SAR) = 180^\circ - (\angle PSD + \angle SAC - \angle IAC) = 180^\circ - (45^\circ + \theta + 90^\circ - \theta - 45^\circ) = 90^\circ
\]
as required. Hence, we get that \(AI = PQ\) and \(AI \perp PQ\).

**Solution 3:** We know that the angle bisector of \(\angle B\) passes through \(P, I\) which implies that \(B, P, I\) are collinear. Similarly, \(C, Q, I\) are also collinear. Since \(I\) is the incentre of \(\triangle ABC\), we know that
\[
\angle PIQ = \angle BIC = 90^\circ + \frac{\angle A}{2} = 135^\circ
\]
Join \(AP, AQ\). We know that \(\angle BAP = \frac{1}{2} \angle BAD = \frac{1}{2} \angle C\). Also, \(\angle ABP = \frac{1}{2} \angle B\). Hence by Exterior Angle Theorem in \(\triangle ABP\), we get that

\[
\angle API = \angle ABP + \angle BAP = \frac{1}{2} (\angle B + \angle C) = 45^\circ 
\]

Similarly in \(\triangle ADC\), we get that \(\angle AQI = 45^\circ\). Also, we have

\[
\angle PAI = \angle BAI - \angle BAP = 45^\circ - \frac{\angle C}{2} = \frac{\angle B}{2} 
\]

Similarly, we get \(\angle QAI = \frac{\angle C}{2}\).

Now applying Sine Rule in \(\triangle API\), we get

\[
\frac{IP}{\sin \angle PAI} = \frac{AI}{\sin \angle API} \implies IP = \sqrt{2} AI \sin \frac{B}{2} 
\]

Similarly, applying Sine Rule in \(\triangle AQI\), we get

\[
\frac{IQ}{\sin \angle PAI} = \frac{AI}{\sin \angle AQI} \implies IQ = \sqrt{2} AI \sin \frac{C}{2} 
\]

Applying Cosine Rule in \(\triangle PIQ\) gives us that

\[
PQ^2 = IP^2 + IQ^2 - 2 \cdot IP \cdot IQ \cos \angle PIQ
\]

We will prove that \((\sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} + \sqrt{2} \sin \frac{B}{2} \sin \frac{C}{2}) = \frac{1}{2}\). In any \(\triangle XYZ\), we have that

\[
\sum_{cyc} \sin^2 \frac{X}{2} = 1 - 2 \prod \sin \frac{X}{2}
\]

Using this in \(\triangle ABC\), and using the fact that \(\angle A = 90^\circ\), we get

\[
\sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} = 1 - 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}
\]

\[
\implies \frac{1}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} = 1 - \sqrt{2} \sin \frac{B}{2} \sin \frac{C}{2}
\]

\[
\implies \left( \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} + \sqrt{2} \sin \frac{B}{2} \sin \frac{C}{2} \right) = \frac{1}{2}
\]

which was to be proved. Hence we get \(PQ = AI\).

The second part of the problem can be obtained by angle-chasing as outlined in Solution 2.

**Solution 4:** Observe that \(\angle APB = \angle AQC = 135^\circ\). Thus \(\angle API = \angle AQI = 45^\circ\) (since \(B - P - I\) and \(C - Q - I\)). Note \(\angle PAQ = 1/2 \angle A = 45^\circ\). Let \(X = BI \cap AQ\) and \(Y = CI \cap AP\). Therefore \(\angle AXP = 180^\circ - \angle API - \angle PAQ = 90^\circ\). Similarly \(\angle AYQ = 90^\circ\).

Hence \(I\) is the orthocentre of triangle \(PAQ\). Therefore \(AI\) is perpendicular to \(PQ\). Also \(AI = 2R_{PAQ} \cos 45^\circ = 2R_{PAQ} \sin 45^\circ = PQ\).
6. Let \( n \geq 1 \) be an integer and consider the sum
\[
x = \sum_{k \geq 0} \binom{n}{2k} 2^{n-2k} 3^k = \left( \binom{n}{0} 2^n + \binom{n}{2} 2^{n-2} \cdot 3 + \binom{n}{4} 2^{n-4} \cdot 3^2 + \cdots \right).
\]
Show that \( 2x - 1, 2x, 2x + 1 \) form the sides of a triangle whose area and inradius are also integers.

**Solution:** Consider the binomial expansion of \( (2 + \sqrt{3})^n \). It is easy to check that
\[
(2 + \sqrt{3})^n = x + y\sqrt{3},
\]
where \( y \) is also an integer. We also have
\[
(2 - \sqrt{3})^n = x - y\sqrt{3}.
\]
Multiplying these two relations, we obtain
\[
x^2 - 3y^2 = 1.
\]
Since all the terms of the expansion of \( (2 + \sqrt{3})^n \) are positive, we see that
\[
x = (2 + \sqrt{3})^n + (2 - \sqrt{3})^n = 2 \left( \binom{n}{0} 2^n + \binom{n}{2} 2^{n-2} \cdot 3 + \cdots \right) \geq 4.
\]
Thus \( x \geq 2 \). Hence \( 2x + 1 < 2x + (2x - 1) \) and therefore \( 2x - 1, 2x, 2x + 1 \) are the sides of a triangle. By Heron’s formula we have
\[
\Delta = 3x(x + 1)(x - 1) = 3x^2(x^2 - 1) = 9x^2y^2.
\]
Hence \( \Delta = 3xy \) which is an integer. Finally, its inradius is
\[
\frac{\text{area}}{\text{perimeter}} = \frac{3xy}{3x} = y,
\]
which is also an integer.

**Solution 2:** We will first show that the numbers \( 2x_n - 1, 2x_n, 2x_n + 1 \) form the sides of a triangle. To show that, it suffices to prove that \( 2x_n - 1 + 2x_n > 2x_n + 1 \). If possible, let the converse hold. Then, we see that we must have \( 4x_n - 1 \leq 2x_n + 1 \), which implies that \( x_n \leq 1 \). But we see that even for the smallest value of \( n = 1 \), we have that \( x_1 > 1 \). Hence, the numbers are indeed sides of a triangle.

Let \( \Delta_n, r_n, s_n \) denote respectively, the area, inradius and semiperimeter of the triangle with sides \( 2x_n - 1, 2x_n, 2x_n + 1 \). By Heron’s Formula for the area of a triangle, we see that
\[
\Delta_n = \sqrt{3x_n(x_n - 1)x_n(x_n + 1)} = x_n \sqrt{3(x_n^2 - 1)}
\]
If possible, let \( \Delta_n \) be an integer for all \( n \in \mathbb{N} \). We see that due to the presence of the first term \( \binom{n}{0} 2^n \), we have \( 3 \mid x_n, \forall n \in \mathbb{N} \). Hence, we get that \( 3 \mid x_n^2 - 1 \). Hence, we can write \( x_n^2 - 1 \) as \( 3m \) for some \( m \in \mathbb{N} \). Then, we can also write
\[
\Delta_n = 3x_n \sqrt{m}
\]
Note that we have assumed that \( \Delta_n \) is an integer. Hence, we see that we must have \( m \) to be a perfect square. Consequently, we get that
\[
\frac{\Delta_n}{s_n} = \frac{\Delta_n}{3x_n} = \sqrt{m} \in \mathbb{Z}
\]
Hence, it only remains to show that $\Delta_n \in \mathbb{Z}$, $\forall n \in \mathbb{N}$. In other words, it suffices to show that $3(x_n^2 - 1)$ is a perfect square for all $n \in \mathbb{N}$.

We see that we can write $x_n$ as

$$x_n = \frac{1}{2} \left( 2 \sum_{k \geq 0} \binom{n}{2k} 2^{n-2k} 3^k \right)$$

$$= \frac{1}{2} \left( (2 + \sqrt{3})^n + (2 - \sqrt{3})^n \right)$$

$$3x_n^2 - 3 = \frac{3}{4} \left( (2 + \sqrt{3})^{2n} + (2 - \sqrt{3})^{2n} + 2(2 + \sqrt{3})^n(2 - \sqrt{3})^n \right) - 3$$

$$= \frac{3}{4} \left( (2 + \sqrt{3})^{2n} + (2 - \sqrt{3})^{2n} - 2(2 + \sqrt{3})^n(2 - \sqrt{3})^n \right)$$

$$= \left( \frac{\sqrt{3}}{2} \left( (2 + \sqrt{3})^n - (2 - \sqrt{3})^n \right) \right)^2$$

We are left to show that the quantity obtained in the above equation is an integer. But we see that if we define

$$a_n = \frac{\sqrt{3}}{2} \left( (2 + \sqrt{3})^n - (2 - \sqrt{3})^n \right), \forall n \in \mathbb{N}$$

the sequence $\langle a_k \rangle_{k=1}^\infty$ thus obtained is exactly the solution for the recursion given by

$$a_{n+2} = 4a_{n+1} - a_n, \forall n \in \mathbb{N}, \ a_1 = 3, a_2 = 12$$

Hence, clearly, each $a_n$ is obviously an integer, thus completing the proof.

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