1. Let $ABC$ be an isosceles triangle with $AB = AC$. Let $\Gamma$ be its circumcircle and let $O$ be the centre of $\Gamma$. Let $CO$ meet $\Gamma$ in $D$. Draw a line parallel to $AC$ through $D$. Let it intersect $AB$ at $E$. Suppose $AE : EB = 2 : 1$. Prove that $ABC$ is an equilateral triangle.

**Solution:** Extend $DE$ to meet $BC$ at $F$. Join $BD$ and $DA$. Since $CD$ is a diameter, we see that $\angle DBC = 90^\circ$. Since $DF$ is parallel to $AC$, it follows that $\triangle EBF \sim \triangle ABC$. Hence $EB = ED$. Since $AE : EB = 2 : 1$ and $EB = ED$, we obtain $AE = 2ED$. Hence $\angle DAB = 30^\circ$. This implies $\angle DCB = 30^\circ$ and hence $\angle BDC = 60^\circ$. But then $\angle BAC = \angle BDC = 60^\circ$ and hence $\triangle ABC$ is equilateral.

2. Let $a, b, c$ be positive real numbers such that

$$\frac{ab}{1 + bc} + \frac{bc}{1 + ca} + \frac{ca}{1 + ab} = 1.$$ 

Prove that $\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} \geq 6\sqrt{2}$.

**Solution:** The given condition is equivalent to

$$\sum ab(1 + ca)(1 + ab) = (1 + ab)(1 + bc)(1 + ca).$$ 

This gives

$$\sum ab + \sum a^2b^2 + abc\sum a + abc\sum a^2b = 1 + \sum ab + abc\sum a + a^2b^2c^2.$$ 

Hence

$$a^2b^2c^2 + 1 = \sum a^2b^2 + abc\sum a^2b.$$ 

Using

$$\sum a^2b^2 \geq 3(ab)^{4/3}, \quad \sum a^2b \geq 3abc,$$

we get

$$a^2b^2c^2 + 1 \geq 3(ab)^{4/3} + 3(ab)^2.$$ 

Taking $x = (abc)^{2/3}$, this reduces to $2x^3 + 3x^2 - 1 \leq 0$. This gives $(x + 1)^2(2x - 1) \leq 0$. Hence $x \leq 1/2$. Therefore $abc \leq 1/2\sqrt{2}$. Finally

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} \geq \frac{3}{abc} \geq 6\sqrt{2}.$$ 

3. The present ages in years of two brothers $A$ and $B$, and their father $C$ are three distinct positive integers $a, b, \text{ and } c\text{ respectively. Suppose }\frac{b - 1}{a - 1} \text{ and } \frac{b + 1}{a + 1} \text{ are two consecutive integers, and } \frac{c - 1}{b - 1} \text{ and } \frac{c + 1}{b + 1} \text{ are two consecutive integers. If } a + b + c \leq 150 \text{ determine } a, b \text{ and } c.$$

**Solution:** We have

$$\frac{b - 1}{a - 1} = l, \quad \frac{b + 1}{a + 1} = l - 1, \quad \frac{c - 1}{b - 1} = m, \quad \frac{c + 1}{b + 1} = m - 1.$$
4. A box contains answer 4032 scripts out of which exactly half have odd number of marks. We choose 2 scripts randomly and, if the scores on both of them are odd number, we add one mark to one of them, put the script back in the box and keep the other script outside. If both scripts have even scores, we put back one of the scripts and keep the other outside. If there is one script with even score and the other with odd score, we put back the script with the odd score and keep the other script outside. After following this procedure a number of times, there are 3 scripts left among which there is at least one script each with odd and even scores. Find, with proof, the number of scripts with odd scores among the three left.

Solution: There are three types of processes. In the first type, the scripts with odd scores decreases by 2. In the second and third types, there is no change in the number of scripts with odd scores. Hence at each step, the number of scripts with odd score decreases by 0 or 2. Since there are 2016 scripts with odd scores, the number of scripts with odd scores at the end is either 0 or 2. Since it is given that there is at least one script with odd scores, two of the three must have odd scores. Hence at each step, the number of scripts with odd score decreases by 0 or 2. Since there are 2016 scripts with odd scores, the number of scripts with odd scores at the end is either 0 or 2. Since it is given that there is at least one script with odd scores, two of the three must have odd scores.

5. Let $ABC$ be a triangle, $AD$ an altitude and $AE$ a median. Assume $B, D, E, C$ lie in that order on the line $BC$. Suppose the incentre of triangle $ABE$ lies on $AD$ and the incentre of $ADC$ lies on $AE$. Find the angles of triangle $ABC$.

Solution: Since $AD \perp BE$ and the incentre of $\triangle ABE$ lies on $AD$, it follows that $ABE$ is isosceles. In particular $\angle BAD = \angle DAE = \alpha$, say. Since $AE$ is the bisector of $\angle DAC$, it follows that $\angle EAC = \angle DAE = \alpha$. Moreover, we have

$$\frac{AD}{AC} = \frac{DE}{CE}.$$

Since $BE = EC = \frac{a}{4}$, we also have $DE = \frac{1}{2}BE = \frac{a}{4}$. Thus we get

$$\frac{AD}{AC} = \frac{DE}{EC} = \frac{a/4}{a/2} = \frac{1}{2}.$$

Since $\triangle ADE$ is a right-angled triangle and $AD = AC/2$, it follows that $\angle ACD = 30^\circ$. Hence $\angle DAC = 60^\circ$. Since $\angle DAC = 2\alpha$, we get $\alpha = 30^\circ$. Now $\angle A = 3\alpha$ and hence $\angle A = 90^\circ$. This gives $\angle B = 60^\circ$.

6. i) Prove that if an infinite sequence of strictly increasing positive integers in arithmetic progression has one cube then it has infinitely many cubes.

(ii) Find, with justification, an infinite sequence of strictly increasing positive integers in arithmetic progression which does not have any cube.

Solution:

i) Let $a$ be the first term of the AP and $d$ be the common difference. (Here $a, d$ are positive integers.) We can find an integer $b$ such that $b^3 = a + (n-1)d$ for some $n \in \mathbb{N}$. Consider $(b+d)^3$. We observe that

$$(b+d)^3 = b^3 + d(3b^2 + 3bd + d^2) = a + (n-1 + 3b^2 + 3bd + d^2)d = a + (m-1)d,$$
where \(m = n + 3b^2 + 3bd + d^2\) is a positive integer. Hence \((b + d)^3\) is also in the given AP. More generally, the same method shows that \((b + kd)^3\) is in the AP for every \(k \in \mathbb{N}\). Hence the given AP contains infinitely many cubes.

ii) Consider the AP \(\langle 2, 6, 10, 14, \ldots \rangle\). Here \(a = 2\) and \(d = 4\). The general term is \(2 + 4k\), where \(k \geq 0\) is an integer. Suppose \(2 + 4k = b^3\) for some integer \(b\). Then 2 divides \(b\). Hence \(b = 2c\) for some \(c\). Therefore \(8c^3 = 2 + 4k\) or \(4c^3 = 2k + 1\). But this is impossible since LHS is even and RHS is odd. We conclude that the AP \(\langle 2, 6, 10, 14, \ldots \rangle\) does not contain any cube.