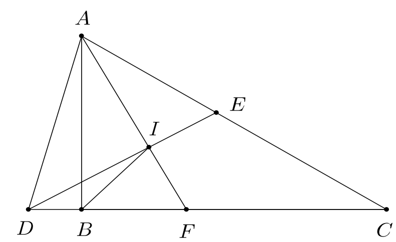
1. Let ABC be a right-angled triangle with $\angle B = 90°$. Let I be the incentre of ABC. Let AI extended intersect BC in F. Draw a line perpendicular to AI at I. Let it intersect AC in E. Prove that $IE = IF$.

Solution: Extend EI to meet CB extended in D. First observe that $ADBI$ is a cyclic quadrilateral since $\angle AID = \angle ABD$. Hence $\angle ADI = \angle ABI = 45°$. Hence $\angle DAI = 45°$. Therefore $IA = ID$.



Consider the triangles AIE and DIF. Both are right triangles. Moreover $\angle IAE = \angle IAB = \angle IDB$. Since $IA = ID$, the triangles are congruent. This means $IE = IF$.

2. Let a, b, c be positive real numbers such that

$$\frac{a}{1+b} + \frac{b}{1+c} + \frac{c}{1+a} = 1.$$

Prove that $abc \leq 1/8$.

Solution: This is equivalent to

$$\sum a(1+c)(1+a) = (1+a)(1+b)(1+c).$$

This simplifies to

$$\sum a^2 + \sum a^2c = 1 + abc$$

Using AM-GM inequality, we have

$$1 + abc = \sum a^2 + \sum a^2c \geq 3(abc)^{2/3} + 3abc.$$

Let $x = (abc)^{1/3}$. Then

$$3x^2 + 2x^3 \leq 1.$$

This can be written as $(x+1)^2(2x-1) \leq 0$. Hence $x \leq 1/2$. Thus

$$abc \leq \frac{1}{8}.$$

3. For any natural number n, expressed in base 10, let $S(n)$ denote the sum of all digits of n. Find all natural numbers n such that $n^3 = 8S(n)^3 + 6nS(n) + 1$.

Solution: We write the given condition as

$$n^3 + (-2S(n))^3 + (-1)^3 = 3 \times n \times (-2S(n)) \times (-1).$$

This is in the form $x^3 + y^3 + z^3 = 3xyz$. We know that this can happen if and only if $x + y + z = 0$. Thus we obtain a simpler condition

$$n - 2S(n) - 1 = 0.$$

Again we know that $n - S(n)$ is divisible by 9. Hence 9 should divide $S(n) + 1$. It is easy to see that the number of digits in n cannot be more than 2. For a three digit number maximum value of $S(n)$ can be 27 and $2S(n) + 1 \leq 55$.. Hence n is either a 1-digit number or a two digit number. Hence $S(n) \leq 18$. Since 9 divides $S(n) + 1$, we can have $S(n) = 8$ or $S(n) = 17$. But then $n = 17$ or $n = 35$. Among these $n = 17$ works but not 35. ($S(35) = 8$ and $2S(n) + 1 = 17 \neq 35$.) Hence the only solution is $n = 17$.

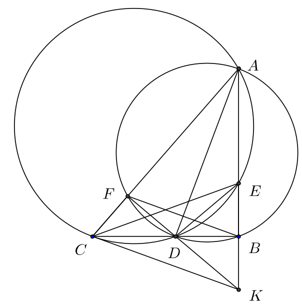
4. How many 6-digit natural numbers containing only the digits 1, 2, 3 are there in which 3 occurs exactly twice and the number is divisible by 9?

   Solution:

   Let $S(n)$ be the sum of the digits of n. Then $n \equiv S(n) \pmod 9$. For any admissible n we observe that $10 \le S(n) \le 14$ and hence there is no value of $S(n)$ that is a multiple of 9. Thus no such n exists.

5. Let ABC be a right-angled triangle with $\angle B = 90°$. Let AD be the bisector of $\angle A$ with D on BC. Let the circumcircle of triangle ACD intersect AB again in E; and let the circumcircle of triangle ABD intersect AC again in F. Let K be the reflection of E in the line BC. Prove that $FK = BC$.

   Solution: First we show that $EB = FC$. Consider the triangles EBD and CFD. Observe that $\angle CFD = 90°$ since $\angle AFD = 90°$ (angle in a semi-circle). Hence $\angle CFD = \angle EBD$. Since $ACDE$ is a cyclic quadrilateral, we have $\angle CDE = 180° - \angle A$. Similarly, we see that $AFDB$ is a cyclic quadrilateral and therefore $\angle FDB = 180° - \angle A$. Thus we obtain $\angle CDE = \angle FDB$. This gives $\angle FDC = \angle BDE$. It follows $\triangle EBD \sim \triangle CFD$.



   Since AD bisects $\angle A$, we have $DB = DF$. Hence $\triangle EBD \cong \triangle CFD$. Hence $FC = EB = BK$. We also observe that $AF = AB$ since $\triangle ABD \cong \triangle AFD$. It follows that $FB \parallel CK$. Since $FC = BK$, we conclude that $CKDF$ is an siosceles trapezium. This gives $FK = BC$.

   Alternate solution: First we show that K, D, F are collinear. Observe that $\angle FDB = 180° - \angle A$ by the concyclicity of $AFDB$. Moreover $\angle BDK = \angle BDE = \angle A$. Therefore $\angle KDF = 180°$. This proves that KDF is a line segment.

   Consider the triangles AKF and ABC. Since both are right-angled triangles and $\angle A$ is common, they are similar. We also see that $\triangle AFD \cong \triangle ABD$ since $\angle AFD = \angle ABD = 90°$, $\angle FAD = \angle BAD = \angle A/2$ and AD common. Hence $AF = AB$. This implies now that $\triangle AFK \cong \triangle ABC$. Hence $KF = BC$.

6. Show that the infinite arithmetic progression $\langle 1, 4, 7, 10, \ldots \rangle$ has infinitely many 3-term subsequences in harmonic progression such that for any two such triples $\langle a_1, a_2, a_3 \rangle$ and $\langle b_1, b_2, b_3 \rangle$ in harmonic progression, one has

$$\frac{a_1}{b_1} \neq \frac{a_2}{b_2} \left(\frac{a_2}{b_2} \neq \frac{a_3}{b_3} \right).$$

   Solution: Consider $\langle 4, 7, 28 \rangle$. We observe that

$$\frac{1}{4} + \frac{1}{28} = \frac{2}{7}.$$

   Thus we look for the terms of the form a, b, ab which give a HP. The condition is

$$\frac{1}{a} + \frac{1}{ab} = \frac{2}{b}.$$

   This reduces to $b(1 + b) = 2ab$ or $2a = 1 + b$. The terms of the given AP are of the form $3k + 1$. If we take $a = 3k + 1$, then $b = 2a - 1 = 6k + 1$. We observe that b is also a term of the gven AP.

Besides, $ab = (3k+1)(6k+1) = 3(6k^2+3k)+1$ is again a term of the given AP. Thus the triple of the form $\langle 3k+1, 6k+1, (3k+1)(6k+1)\rangle$ form a HP. We observe that

$$\frac{3k+1}{3l+1} \neq \frac{6k+1}{6l+1} \neq \frac{(3k+1)(6k+1)}{(3l+1)(6l+1)}.$$

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