Solutions to problems of RMO 2014 (Mumbai region)

1. Three positive real numbers \(a, b, c\) are such that \(a^2 + 5b^2 + 4c^2 - 4ab - 4bc = 0\). Can \(a, b, c\) be the lengths of the sides of a triangle? Justify your answer.

Solution

No. Note that
\[
a^2 + 5b^2 + 4c^2 - 4ab - 4bc = (a - 2b)^2 + (b - 2c)^2 = 0 \Rightarrow a : b : c = 4 : 2 : 1 \Rightarrow b + c : a = 3 : 4.
\]
The triangle inequality is violated.

2. The roots of the equation
\[x^3 - 3ax^2 + bx + 18c = 0\]
form a non-constant arithmetic progression and the roots of the equation
\[x^3 + bx^2 + x - c^3 = 0\]
form a non-constant geometric progression. Given that \(a, b, c\) are real numbers, find all positive integral values \(a\) and \(b\).

Solution

Let \(\alpha - d, \alpha, \alpha + d (d \neq 0)\) be the roots of the first equation and let \(\beta/r, \beta, \beta r (r > 0\) and \(r \neq 1)\) be the roots of the second equation. It follows that \(\alpha = a, \beta = c\) and
\[
a^3 - ad^2 = -18c; \quad 3a^2 - d^2 = b, \tag{1}
\]
\[
c(1/r + 1 + r) = -b; \quad c^2(1/r + 1 + r) = 1. \tag{2}
\]
Eliminating \(d, r\) and \(c\) yields
\[
ab^2 - 2a^3b - 18 = 0, \tag{3}
\]
whence \(b = a^2 \pm (1/a)\sqrt{a^6 + 18a}\). For positive integral values of \(a\) and \(b\) it must be that \(a^6 + 18a\) is a perfect square. Let \(x^2 = a^6 + 18a\). Then \(a^3 < x^2 < a^3 + 1\) for \(a > 2\) and hence no solution. For \(a = 1\) there is no solution. For \(a = 2, x = 10\) and \(b = 9\). Thus the admissible pair is \((a, b) = (2, 9)\).

3. Let \(ABC\) be an acute-angled triangle in which \(\angle ABC\) is the largest angle. Let \(O\) be its circumcentre. The perpendicular bisectors of \(BC\) and \(AB\) meet \(AC\) at \(X\) and \(Y\) respectively. The internal bisectors of \(\angle AXB\) and \(\angle BYC\) meet \(AB\) and \(BC\) at \(D\) and \(E\) respectively. Prove that \(BO\) is perpendicular to \(AC\) if \(DE\) is parallel to \(AC\).

Solution

Observe that triangles \(AYB\) and \(BXC\) are isosceles \((AY = BY\) and \(BX = CX\)). This implies \(\angle BYC = 2\angle BAC\) and \(\angle AXB = 2\angle ACB\). Since \(XD\) and \(YE\) are angle bisectors we have \(\angle AXD = \angle ACB\) and \(\angle CYE = \angle CAB\). Hence \(XD\) is parallel to \(BC\) and \(YE\) is parallel to \(AB\). Therefore
\[
\frac{CE}{EB} = \frac{CY}{AY}, \tag{4}
\]
and
\[
\frac{AD}{DB} = \frac{AX}{CX}. \tag{5}
\]
Now, if $DE$ is parallel to $AC$ then $\frac{CE}{EB} = \frac{AD}{DB}$. Therefore we must have

$$\frac{CY}{AY} = \frac{AX}{CX}. \quad (6)$$

But then

$$\frac{CY}{AY} + 1 = \frac{AX}{CX} + 1 \Rightarrow \frac{AC}{AY} = \frac{AC}{CX} \Rightarrow AY = CX. \quad (7)$$

Hence $BY = AY = CX = BX$. Thus $\angle BXY = \angle BYX$ i.e. $\angle AXB = \angle BYC$ or $\angle ACB = \angle BAC$ i.e triangle $ABC$ is isosceles with $AB = CB$. Hence $BO$ is the perpendicular bisector of $AC$.

4. A person moves in the $x-y$ plane moving along points with integer co-ordinates $x$ and $y$ only. When she is at point $(x,y)$, she takes a step based on the following rules:

(a) if $x+y$ is even she moves to either $(x+1,y)$ or $(x+1,y+1)$;
(b) if $x+y$ is odd she moves to either $(x,y+1)$ or $(x+1,y+1)$.

How many distinct paths can she take to go from $(0,0)$ to $(8,8)$ given that she took exactly three steps to the right ($(x,y)$ to $(x+1,y)$)?

**Solution**

We note that she must also take three up steps and five diagonal steps. Now, a step to the right or an upstep changes the parity of the co-ordinate sum, and a diagonal step does not change it. Therefore, between two right steps there must be an upstep and similarly between two upsteps there must be a right step. We may, therefore write

$$HVHVHV$$

The diagonal steps may be distributed in any fashion before, in between and after the HV sequence. The required number is nothing but the number of ways of distributing 5 identical objects into 7 distinct boxes and is equal to $\binom{11}{6}$.

5. Let $a$, $b$, $c$ be positive numbers such that

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} \leq 1.$$

Prove that $(1 + a^2)(1 + b^2)(1 + c^2) \geq 125$. When does the equality hold?

**Solution**

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} \leq 1 \Rightarrow \frac{a}{1+a} \geq \frac{1}{1+b} + \frac{1}{1+c}. \quad (8)$$

Similarly,

$$\frac{b}{1+b} \geq \frac{1}{1+a} + \frac{1}{1+c}, \quad \frac{c}{1+c} \geq \frac{1}{1+a} + \frac{1}{1+b}. \quad (9)$$

Apply AM-GM to get that

$$\frac{a}{1+a} \geq \frac{2}{\sqrt{(1+b)(1+c)}}, \quad \frac{b}{1+b} \geq \frac{2}{\sqrt{(1+a)(1+c)}}, \quad \frac{c}{1+c} \geq \frac{2}{\sqrt{(1+a)(1+b)}}. \quad (10)$$
Multiplying these results we get
\[ abc \geq 8. \quad (11) \]

Now take
\[ F = (1 + a^2)(1 + b^2)(1 + c^2) \geq 1 + a^2 + b^2 + c^2 + a^2b^2 + b^2c^2 + c^2a^2 + a^2b^2c^2 \quad (12) \]
and apply AM-GM to \( a^2, b^2, c^2 \) and to \( a^2b^2, b^2c^2, c^2a^2 \) to get
\[ F \geq 1 + 3(a^2b^2c^2)^{1/3} + 3(a^4b^4c^4)^{1/3} + a^2b^2c^2 = [1 + (a^2b^2c^2)^{1/3}]^3 \geq [1 + 8^{2/3}]^3 = 125. \quad (13) \]
Wherein the equality holds when \( a = b = c = 2 \).

6. Let \( D, E, F \) be the points of contact of the incircle of an acute-angled triangle \( ABC \) with \( BC, CA, AB \) respectively. Let \( I_1, I_2, I_3 \) be the incentres of the triangles \( AFE, BDF, CED \), respectively. Prove that the lines \( I_1D, I_2E, I_3F \) are concurrent.

Solution

Observe that \( \angle AFE = \angle AEF = 90^\circ - A/2 \) and \( \angle FDE = \angle AEF = 90^\circ - A/2 \). Again \( \angle EI_1F = 90^\circ + A/2 \). Thus
\[ \angle EI_1F + \angle FDE = 180^\circ. \]
Hence \( I_1 \) lies on the incircle. Also
\[ \angle I_1FE = (1/2)\angle AFE = (1/2)\angle AEF = \angle I_1EF. \quad (14) \]
Thus \( I_1E = I_1F \). But then they are equal chords of a circle and so they must subtend equal angles at the circumference. Therefore \( \angle I_1DF = \angle I_1DE \) and so \( I_1D \) is the internal bisector of \( \angle FDE \). Similarly we can show that \( I_2E \) and \( I_3F \) are internal bisectors of \( \angle DEF \) and \( \angle DFE \) respectively. Thus the three lines \( I_1D, I_2E, I_3F \) are concurrent at the incentre of triangle \( DEF \).