Solutions to RMO-2014 problems

1. Let $ABC$ be an acute-angled triangle and suppose $\angle ABC$ is the largest angle of the triangle. Let $R$ be its circumcentre. Suppose the circumcircle of triangle $ARB$ cuts $AC$ again in $X$. Prove that $RX$ is perpendicular to $BC$.

Solution: Extend $RX$ to meet $BC$ in $E$. We show that $\angle XEC = 90^\circ$. Join $RA$, $RB$ and $BX$. Observe that $\angle AXB = \angle ARB = 2\angle C$ and $\angle BXR = \angle BAR = 90^\circ - \angle C$. Hence $\angle EXC = 180^\circ - 2\angle C - (90^\circ - \angle C) = 90^\circ - \angle C$. This shows that $\angle CEX = 90^\circ$.

2. Find all real numbers $x$ and $y$ such that

$$x^2 + 2y^2 + \frac{1}{2} \leq x(2y + 1).$$

Solution: We write the inequality in the form

$$2x^2 + 4y^2 + 1 - 4xy - 2x \leq 0.$$

Thus $(x^2 - 4xy + 4y^2) + (x^2 - 2x + 1) \leq 0$. Hence

$$(x - 2y)^2 + (x - 1)^2 \leq 0.$$

Since $x, y$ are real, we know that $(x - 2y)^2 \geq 0$ and $(x - 1)^2 \geq 0$. Hence it follows that $(x - 2y)^2 = 0$ and $(x - 1)^2 = 0$. Therefore $x = 1$ and $y = \frac{1}{2}$.

3. Prove that there does not exist any positive integer $n < 2310$ such that $n(2310 - n)$ is a multiple of 2310.

Solution: Suppose there exists $n$ such that $0 < n < 2310$ and $n(2310 - n) = 2310k$. Then $n^2 = 2310(n - k)$. But $2310 = 2 \times 3 \times 5 \times 7 \times 11$, the product of primes. Hence $n - k = 2310l^2$ for some $l$. But $n < 2310$ and hence $n - k < 2310$. Hence $l = 0$. This forces $n = k$ and hence $n^2 = 2310(n - k) = 0$. Thus $n = 0$ and we have a contradiction.

4. Find all positive real numbers $x, y, z$ such that

$$2x - 2y + \frac{1}{z} = \frac{1}{2014}, \quad 2y - 2z + \frac{1}{x} = \frac{1}{2014}, \quad 2z - 2x + \frac{1}{y} = \frac{1}{2014}.$$

Solution: Adding the three equations, we get

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{3}{2014}.$$
We can also write the equations in the form

\[
2zx - 2zy + 1 = \frac{z}{2014}, \quad 2xy - 2xz + 1 = \frac{x}{2014}, \quad 2yz - 2yx + 1 = \frac{y}{2014}.
\]

Adding these, we also get

\[2014 \times 3 = x + y + z.\]

Therefore

\[
\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)(x + y + z) = \frac{3}{2014} \times (2014 \times 3) = 9.
\]

Using AM-GM inequality, we therefore obtain

\[9 = \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)(x + y + z) \geq 9 \times (xyz)^{1/3} \left(\frac{1}{xyz}\right)^{1/3} = 9.\]

Hence equality holds in AM-GM inequality and we conclude \(x = y = z\). Thus

\[\frac{1}{x} = \frac{1}{2014},\]

which gives \(x = 2014\). We conclude

\[x = 2014, \quad y = 2014, \quad z = 2014.\]

5. Let \(ABC\) be a triangle. Let \(X\) be on the segment \(BC\) such that \(AB = AX\). Let \(AX\) meet the circumcircle \(\Gamma\) of triangle \(ABC\) again at \(D\). Show that the circumcentre of \(\triangle BDX\) lies on \(\Gamma\).

**Solution:** Draw perpendicular from \(A\) to \(BC\) and extend it to meet \(\Gamma\) in \(F\). We show that \(F\) is the circumcentre of \(\triangle BDX\). Since \(AB = AX\), we observe that \(F\) lies on the perpendicular bisector of \(BX\). Join \(CF\) and \(CD\). We observe that \(\angle ABX = \angle CDX\) and \(\angle AXB = \angle CXD\). Hence \(\triangle ABX\) is similar to \(\triangle CDX\). In particular \(\triangle CDX\) is isosceles.

Moreover, \(\angle BCF = \angle BAF\) and \(\angle DCF = \angle DAF\). Since \(AF\) is the perpendicular bisector of \(BX\), it also bisects \(\angle BAX\). It follows that \(CF\) bisects \(\angle DCX\) and hence \(F\) lies on the perpendicular bisector of \(DX\). Together \(F\) is the circumcentre of \(\triangle BXD\).

6. For any natural number \(n\), let \(S(n)\) denote the sum of the digits of \(n\). Find the number of all 3-digit numbers \(n\) such that \(S(S(n)) = 2\).

**Solution:** Observe that \(S(S(n)) = 2\) implies that \(S(n) = 2, 11\) or \(20\). Hence we have to find the number of all 3 digit numbers \(abc\) such that \(a + b + c = 2, 11\)
and 20. In fact we can enumerate all these:

\[
a + b + c = 2: \quad \overline{abc} = 101, 110, 200;
\]

\[
\]

\[
\]

There are totally 85 three digit numbers having second digital sum equal to 2.