Solutions to RMO-2014 problems

1. Let $ABC$ be a triangle and let $AD$ be the perpendicular from $A$ on to $BC$. Let $K, L, M$ be points on $AD$ such that $AK = KL = LM = MD$. If the sum of the areas of the shaded regions is equal to the sum of the areas of the unshaded regions, prove that $BD = DC$.

**Solution:** let $BD = 4x$, $DC = 4y$ and $AD = 4h$. Then the sum of the areas of the shaded regions is
\[
\frac{1}{2} h \left( x + (y + 2y) + (2x + 3x) + (3y + 4y) \right) = \frac{h(6x + 10y)}{2}.
\]
The sum of the areas of the unshaded regions is
\[
\frac{1}{2} h \left( y + (x + 2x) + (2y + 3y) + (3x + 4x) \right) = \frac{h(10x + 6y)}{2}.
\]
Therefore the given condition implies that
\[
6x + 10y = 10x + 6y.
\]
This gives $x = y$. Hence $BD = DC$.

2. Let $a_1, a_2, \ldots, a_{2n}$ be an arithmetic progression of positive real numbers with common difference $d$. Let
(i) $a_1^2 + a_3^2 + \cdots + a_{2n-1}^2 = x$,  
(ii) $a_2^2 + a_4^2 + \cdots + a_{2n}^2 = y$, and
(iii) $a_n + a_{n+1} = z$.
Express $d$ in terms of $x, y, z, n$.

**Solution:** Observe that
\[
y - x = (a_2^2 - a_1^2) + (a_4^2 - a_3^2) + \cdots + (a_{2n}^2 - a_{2n-1}^2).
\]
The general difference is
\[
a_{2k}^2 - a_{2k-1}^2 = (a_{2k} + a_{2k-1})d = (2a_1 + (2k - 1) + (2k - 2))d.
\]
Therefore
\[
y - x = (2na_1 + (1 + 2 + 3 + \cdots + (2n - 1))d) = nd(2a_1 + (2n - 1)d).
\]
We also observe that
\[
z = a_n + a_{n+1} = 2a_1 + (2n - 1)d.
\]
It follows that $y - x = ndz$. Hence $d = (y - x)/nz$. 
3. Suppose for some positive integers \( r \) and \( s \), the digits of \( 2^r \) is obtained by permuting the digits of \( 2^s \) in decimal expansion. Prove that \( r = s \).

**Solution:** Suppose \( s \leq r \). If \( s < r \) then \( 2^s < 2^r \). Since the number of digits in \( 2^s \) and \( 2^r \) are the same, we have \( 2^r < 10 \times 2^s < 2^{s+4} \). Thus we have \( 2^s < 2^r < 2^{s+4} \) which gives \( r = s + 1 \) or \( s + 2 \) or \( s + 3 \). Since \( 2^r \) is obtained from \( 2^s \) by permuting its digits, \( 2^r - 2^s \) is divisible by 9. If \( r = s + 1 \), we see that \( 2^r - 2^s = 2^s \) and it is clearly not divisible by 9. Similarly, \( 2^{s+2} - 2^s = 3 \times 2^s \) and \( 2^{s+3} - 2^s = 7 \times 2^s \) and none of these is divisible by 9. We conclude that \( s < r \) is not possible. Hence \( r = s \).

4. Is it possible to write the numbers 17, 18, 19, \ldots, 32 in a 4 \times 4 grid of unit squares, with one number in each square, such that the product of the numbers in each 2 \times 2 sub-grids \( AMRG, GRND, MBHR \) and \( RHCN \) is divisible by 16?

**Solution:** NO! If the product in each 2 \times 2 sub-square is divisible by 16, then the product of all the numbers is divisible by \( 16 \times 16 \times 16 \times 16 = 2^{16} \). But it is easy to see that

\[
17 \times 18 \times 19 \times \cdots \times 32 = 2^{15}k,
\]

where \( k \) is an odd number. Hence the product of all the numbers in the grid is not divisible by \( 2^{16} \).

5. Let \( ABC \) be an acute-angled triangle and let \( H \) be its ortho-centre. For any point \( P \) on the circum-circle of triangle \( ABC \), let \( Q \) be the point of intersection of the line \( BH \) with the line \( AP \). Show that there is a unique point \( X \) on the circum-circle of \( ABC \) such that for every point \( P \neq A, B \), the circum-circle of \( HQP \) pass through \( X \).

**Solution:** We consider two possibilities: \( Q \) lying between \( A \) and \( P \); and \( P \) lying between \( A \) and \( Q \). (See the figures.)

In the first case, we observe that

\[
\angle HXC = \angle HXP + \angle PXC = \angle AQB + \angle PAC,
\]

since \( Q, H, X, P \) are concyclic and \( P, A, X, C \) are also concyclic. Thus we get

\[
\angle HXC = \angle AQE + \angle QAE = 90^\circ
\]

because \( BE \perp AC \).
In the second case, we have
\[ \angle HXC = \angle HXP + \angle PXC = \angle HQP + \angle PAC; \]
the first follows from \( H, X, Q, P \) are concyclic; the second follows from the concyclic-
ity of \( A, X, C, P \). Again \( BE \perp AC \) shows that \( \angle HXC = 90^\circ \).

Thus for any point \( P \neq A, B \) on the circumcircle of \( ABC \), the point \( X \) of intersection
of the circumcircles of \( ABC \) and \( HPQ \) is such that \( \angle HXC = 90^\circ \). This means \( X \)
is precisely the point of intersection of the circumcircles of \( HEC \) and \( ABC \), which
is independent of \( P \).

6. Let \( x_1, x_2, \ldots, x_{2014} \) be positive real numbers such that \( \sum_{j=1}^{2014} x_j = 1 \). Determine
with proof the smallest constant \( K \) such that
\[ K \sum_{j=1}^{2014} \frac{x_j^2}{1 - x_j} \geq 1. \]

**Solution:** Let us take the general case: \( \{x_1, x_2, \ldots, x_n\} \) are positive real numbers
such that \( \sum_{k=1}^{n} x_k = 1 \). Then
\[ \sum_{k=1}^{n} \frac{x_k^2}{1 - x_k} = \sum_{k=1}^{n} \frac{x_k^2 - 1}{1 - x_k} + \sum_{k=1}^{n} \frac{1}{1 - x_k} = \sum_{k=1}^{n} (-1 - x_k) + \sum_{k=1}^{n} \frac{1}{1 - x_k}. \]
Now the first sum is \(-n - 1\). We can estimate the second sum using AM-HM
inequality:
\[ \sum_{k=1}^{n} \frac{1}{1 - x_k} \geq \frac{n^2}{\sum_{k=1}^{n} (1 - x_k)} = \frac{n^2}{n - 1}. \]

Thus we obtain
\[ \sum_{k=1}^{n} \frac{x_k^2}{1 - x_k} \geq -(1 + n) + \frac{n^2}{n - 1} = \frac{1}{n - 1}. \]
Here equality holds if and only if all \( x_j \)’s are equal. Thus we get the smallest
constant \( K \) such that
\[ K \sum_{j=1}^{2014} \frac{x_j^2}{1 - x_j} \geq 1 \]
to be \( 2014 - 1 = 2013 \).