Model solutions of RMO 2012 (Mumbai region)

1. Let α be the common zero of the three given polynomials. Then

$$\alpha^2 + a\alpha + b = 0; \tag{1}$$

$$\alpha^2 + \alpha + ab = 0; \tag{2}$$

$$a\alpha^2 + \alpha + b = 0. \tag{3}$$

(1)-(2) yields

$$(a-1)(\alpha - b) = 0. \tag{4}$$

(3)-(2) yields

$$(a-1)(\alpha^2 - b) = 0. \tag{5}$$

From (4) we conclude that $a = 1$ or $\alpha = b$ and from (5) we see that $a = 1$ or $\alpha^2 = b$. But if $a = 1$ then the three polynomials are same as $x^2 + x + b$, contradicting the fact that they are different. Therefore we must have $\alpha = b$ and $\alpha^2 = b$. Thus $\alpha = \alpha^2$ i.e $\alpha = 0$ or $\alpha = 1$. If $\alpha = 0$ then $b = 0$ and this is not a feasible solution since we need to find $b \neq 0$. Hence $\alpha = 1$, which yields $b = 1$ and from (1), $a = -2$.
Thus $a = -2$, $b = 1$ and the polynomials are $x^2 - 2x + 1$, $x^2 + x - 2$ and $-2x^2 + x + 1$.

2. Observe that $n^2 + 3n + 51 = (n-5)(n+8) + 91$. If $13|n^2 + 3n + 51$, then $13|(n-5)(n+8)$. Therefore $13|n-5$ or $13|n+8$. Observe that

$$13|n - 5 \Leftrightarrow 13|(n+8) - 13 \Leftrightarrow 13|n+8;$$

Now, $21n^2 + 89n + 44 = (7n+4)(3n+11) = \{(3(n+8) + 4(n-5)\}\{2(n+8) + (n-5)\}$. Writing $n + 8 = 13m_1$ and $n - 5 = 13m_2$, where m_1 and m_2 are positive integers we get

$$21n^2 + 89n + 44 = 169(3m_1 + 4m_2)(2m_1 + m_2).$$

Therefore 169 divides $21n^2 + 89n + 44$.

Comments:

There were different solutions to this problem by the students. We present two such solutions.

First solution:

Observe that $n^2 + 3n + 51 \equiv 0 \equiv \{(n-5)^2 + 26\}(\text{mod } 13) \Rightarrow n \equiv 5(\text{mod } 13)$. Now $21n^2 + 89n + 44 = (7n+4)(3n+11) \equiv 0(\text{mod}169)$ because $7n + 4 \equiv 39 \equiv 0(\text{mod}13)$ and $3n + 11 \equiv 26 \equiv 0(\text{mod}13)$.

Second solution:

$13|n^2 + 3n + 51 \Rightarrow 13|(n+8)(n-5) \Rightarrow n \equiv 5(\text{mod}13)$. Let $n_k = 13k + 5$, k an integer, and $f(n_k) = 21n_k^2 + 89n_k + 44$. Then

$$f(n_{k+1}) - f(n_k) = 169(44 + 42k) \equiv 0(\text{mod}169) \Rightarrow f(n_k) \equiv f(n_0)(\text{mod } 169).$$

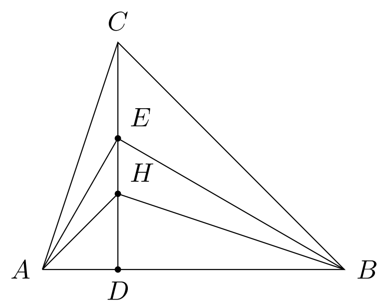
As $f(n_0) = 1014 \equiv 0(\text{mod}169)$ we conclude that $f(n_k) \equiv 0(\text{ mod}169)$.

3. Writing $x = [x] + \{x\}$ in the given equation and simplifying we obtain $2^{5\{x\}} = \dfrac{5.2^{2[x]}}{2^{2[x]} - 11}$. As $0 \leq \{x\} < 1$, $1 \leq 2^{5\{x\}} < 32$ and hence the right hand side is positive. Therefore $2^{2[x]} - 11 > 0$, i.e $[x] \geq 2$. Also, $5 < \dfrac{5.2^{2[x]}}{2^{2[x]} - 11} \leq 16$. Hence there is a solution for every real x with $[x] \geq 2$ of which only one is rational, namely $x = 14/5$.

4. Since $AEB = 90°$ and ED is perpendicular to AB, $ED^2 = AD.DB$. Now

$$[ABE]^2 = [ABC][ABH] \Leftrightarrow ED^2 = HD.CD \Leftrightarrow AD.DB = HD.CD.$$

But triangles ADH and CDB are similar because $\angle DAH = \angle BCD$, $\angle ADH = \angle CDB$.



Therefore, $AD/CD = HD/DB$, i.e $AD.DB = HD.CD$. Thus $[ABE]^2 = [ABC][ABH]$.

5. We have $\dfrac{1}{a} + \dfrac{2}{b} + \dfrac{3}{c} = 1$ and $1 \leq a \leq b \leq c$. Therefore $1/a \geq 1/b \geq 1/c$ and

$$1 = \frac{1}{a} + \frac{2}{b} + \frac{3}{c} \leq \frac{1}{a} + \frac{2}{a} + \frac{3}{a} = \frac{6}{a} \Rightarrow a \leq 6.$$

Since a is a prime and $1 \leq a \leq 6$, the possible values of a are 2, 3 and 5.

Case 1: $a = 2$.

$a = 2 \Rightarrow \dfrac{2}{b} + \dfrac{3}{c} = \dfrac{1}{2}$. Now $\dfrac{2}{b} < \dfrac{1}{2} \Rightarrow b \geq 5$ and $1/b \geq 1/c \Rightarrow \dfrac{1}{2} = \dfrac{2}{b} + \dfrac{3}{c} \leq \dfrac{5}{b} \Rightarrow b \leq 10$. Hence $5 \leq b \leq 10$. Substituting the possible values of b in the equation $\dfrac{2}{b} + \dfrac{3}{c} = \dfrac{1}{2}$ we obtain $(b, c) = (5, 30), (6, 18), (7, 14), (8, 12), (10, 10)$ as the admissible pairs. Therefore the solutions in this case are $(a, b, c) = (2, 5, 30), (2, 6, 18), (2, 7, 14), (2, 8, 12), (2, 10, 10)$.

Case 2: $a = 3$.

Emulating the method outlined in the analysis of **Case 1** we find that $4 \leq b \leq 7$ and the solutions in this case are $(a, b, c) = (3, 4, 18), (3, 6, 9)$.

Case 3: $a = 5$.

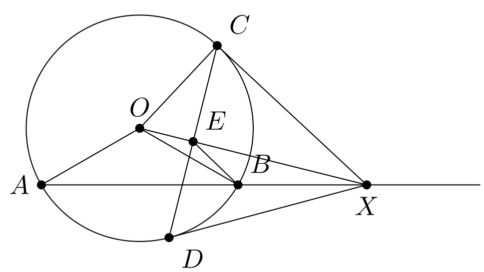
There is no solution in this case.
In summary, the solutions are $(a, b, c) = (2, 5, 30), (2, 6, 18), (2, 7, 14), (2, 8, 12), (2, 10, 10), (3, 4, 18), (3, 6, 9)$.

6. WSUM = 70656. Partition the set of all subsets of S into two sets -T1- consisting of those subsets which contain the element 1 and T2 those which do not contain 1. For every subset S2 belonging to T2, there is a unique subset S1 belonging to T1 which is $\{1\} \cup S2$. Consider any element $a \geq 2$ of S. Let S2 be such that the element a occupies an even numbered position in it. It will occupy an odd numbered position in S_1. The total contribution to WSUM by a from both these subsets is $5a$. The same is true if a occpies an odd numbered position in S_2. Therefore, the total contribution of the element a to WSUM is $5a$ multiplied by the number of subsets of T_2 that contain a. The number of these subsets is 2^8. The contribution of the element 1 to WSUM is clearly 3×2^9. Therefore the sum of all WSUMs is

$$T = 3 \times 2^9 + 5 \times 2^8 \times \sum_{j=2}^{10} j = 276 \times 2^8 = 70656.$$

Aliter: The total contribution of the element $a \geq 2$ over all subsets is $t = \sum_{j=0}^{a-1} \binom{a-1}{j}(2 + \xi(j))2^{10-a}a = 0$. Here $\xi(j) = 0$ if j is odd and $\xi(j) = 1$ if j is even. $\binom{a-1}{j}$ is the number of j-element subsets of $\{1, 2, ..., a-1\}$ and 2^{10-a} is the total number of subsets of $\{a+1, a+2, ..., 10\}$. Simplifying, we get $t = 2.2^9a + 2^8a = 5.2^8a$.

7. O, E, and X are collinear. Join O with A, B and C. Triangles OCX and CEX are similar.



Therefore $XC/XO = XE/XC$, i.e $XC^2 = XO.XE$. But $XC^2 = XB.XA$. Hence $XB.XA = XE.XO$ implying B, A, O, E are concyclic. Therefore $\angle OAB = 180° - \angle OEB = 40°$. So, $\angle AOB = 180° - 2\angle OAB = 100°$.

8. Put $x = 2a$, $y = 2b$ and $z = 2c$. The problem reduces to showing

$$\frac{1}{(a-1)(b-1)(c-1)} + \frac{8}{(a+1)(b+1)(c+1)} \leq \frac{1}{4}$$

subject to $1/a+1/b+1/c = 1$. Observe that $a > 1, b > 1, c > 1$ and $a-1 = a\left(\frac{1}{b} + \frac{1}{c}\right) \geq \frac{2a}{\sqrt{bc}}$ (by A.M-G.M inequality). Similarly we get $b-1 \geq \frac{2b}{\sqrt{ca}}$ and $c-1 \geq \frac{2c}{\sqrt{ab}}$. Multiplying these and taking the reciprocal we obtain

$$\frac{1}{(a-1)(b-1)(c-1)} \leq \frac{1}{8}. \quad \ldots \text{(I)}$$

Next observe that $\frac{a+1}{a-1} = 1 + \frac{2}{a-1} \geq \frac{2\sqrt{2}}{\sqrt{a-1}}$ whence $a+1 \geq 2\sqrt{2(a-1)}$. Similarly we obtain $b+1 \geq 2\sqrt{2(b-1)}$ and $c+1 \geq 2\sqrt{2(c-1)}$. Multiplying these yields

$$(a+1)(b+1)(c+1) \geq 16\sqrt{2(a-1)(b-1)(c-1)} \geq 16\sqrt{2.8} = 64.$$

Therefore

$$\frac{8}{(a+1)(b+1)(c+1)} \leq \frac{1}{8}. \quad \ldots \text{(II)}$$

By adding (I) and (II) we get

$$\frac{1}{(a-1)(b-1)(c-1)} + \frac{8}{(a+1)(b+1)(c+1)} \leq \frac{1}{4}.$$

Comments:

We present another method which many students had adopted. By the A.M-G.M-H.M inequality,

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc} \geq \frac{3}{1/a+1/b+1/c} = 3.$$

Thus $a+b+c \geq 9$ and $abc \geq 27$. Using these two inequalities we obtain

$$(a-1)(b-1)(c-1) = abc - (ab+bc+ca) + (a+b+c) - 1 \geq 8,$$
$$(a+1)(b+1)(c+1) = abc+(ab+bc+ca)+(a+b+c)+1 = 2abc+(a+b+c)+1 \geq 2(27)+9+1 = 64.$$

From these two inequalities we get

$$\frac{1}{(a-1)(b-1)(c-1)} + \frac{8}{(a+1)(b+1)(c+1)} \leq \frac{1}{4}.$$