Model solutions of RMO 2012 (Mumbai region)

1. Let $\alpha$ be the common zero of the three given polynomials. Then

$$\alpha^2 + ab + b = 0; \quad (1)$$
$$\alpha^2 + \alpha + ab = 0; \quad (2)$$
$$aa^2 + a + b = 0. \quad (3)$$

(1)-(2) yields

$$(a - 1)(\alpha - b) = 0. \quad (4)$$

(3)-(2) yields

$$(a - 1)(\alpha^2 - b) = 0. \quad (5)$$

From (4) we conclude that $a = 1$ or $\alpha = b$ and from (5) we see that $a = 1$ or $\alpha^2 = b$. But if $a = 1$ then the three polynomials are same as $x^2 + x + b$, contradicting the fact that they are different. Therefore we must have $\alpha = b$ and $\alpha^2 = b$. Thus $\alpha = \alpha^2$ i.e $\alpha = 0$ or $\alpha = 1$. If $\alpha = 0$ then $b = 0$ and this is not a feasible solution since we need to find $b \neq 0$. Hence $\alpha = 1$, which yields $b = 1$ and from (1), $a = -2$.

Thus $a = -2$, $b = 1$ and the polynomials are $x^2 - 2x + 1$, $x^2 + x - 2$ and $-2x^2 + x + 1$.

2. Observe that $n^2 + 3n + 51 = (n - 5)(n + 8) + 91$. If $13|n^2 + 3n + 51$, then $13|(n - 5)(n + 8)$. Therefore $13|n - 5$ or $13|n + 8$. Observe that

$$13|n - 5 \iff 13|(n + 8) \iff 13|n + 8;$$

Now, $21n^2 + 89n + 44 = (7n + 4)(3n + 11) = \{(3(n + 8) + 4(n - 5))\{2(n + 8) + (n - 5)\}$. Writing $n + 8 = 13m_1$ and $n - 5 = 13m_2$, where $m_1$ and $m_2$ are positive integers we get

$$21n^2 + 89n + 44 = 169(3m_1 + 4m_2)(2m_1 + m_2).$$

Therefore 169 divides $21n^2 + 89n + 44$.

Comments:

There were different solutions to this problem by the students. We present two such solutions.

First solution:

Observe that $n^2 + 3n + 51 \equiv 0 \equiv \{(n - 5)^2 + 26\} (mod 13) \Rightarrow n \equiv 5(mod 13)$. Now $21n^2 + 89n + 44 = (7n + 4)(3n + 11) \equiv 0(mod 169)$ because $7n + 4 \equiv 39 \equiv 0(mod13)$ and $3n + 11 \equiv 26 \equiv 0(mod13)$.

Second solution:

$13|n^2 + 3n + 51 \Rightarrow 13|(n + 8)(n - 5) \Rightarrow n \equiv 5(mod13)$. Let $n_k = 13k + 5$, $k$ an integer, and $f(n_k) = 21n_k^2 + 89n_k + 44$. Then

$$f(n_{k+1}) - f(n_k) = 169(44 + 42k) \equiv 0(mod169) \Rightarrow f(n_k) \equiv f(n_0)(mod 169).$$

As $f(n_0) = 1014 \equiv 0(mod169)$ we conclude that $f(n_k) \equiv 0(\ mod169)$. 
3. Writing \( x = [x] + \{x\} \) in the given equation and simplifying we obtain
\[
2^{5\{x\}} = \frac{5.2^{2[x]}}{2^{2[x]} - 11}.
\]
As \( 0 \leq \{x\} < 1 \), \( 1 \leq 2^{5\{x\}} < 32 \) and hence the right hand side is positive. Therefore
\[
2^{2[x]} - 11 > 0, \text{ i.e } [x] \geq 2.
\]
Also, \( 5 < \frac{5.2^{2[x]}}{2^{2[x]} - 11} \leq 16 \). Hence there is a solution for every real \( x \) with \( [x] \geq 2 \) of which only one is rational, namely \( x = 14/5 \).

4. Since \( \angle AEB = 90^\circ \) and \( \overline{ED} \) is perpendicular to \( \overline{AB} \),
\[
[\overline{ABE}]^2 = [\overline{ABC}][\overline{ABH}] \iff \overline{ED}^2 = \overline{HD} \cdot \overline{CD} \iff \overline{AD} \cdot \overline{DB} = \overline{HD} \cdot \overline{CD}.
\]
But triangles \( \triangle ADH \) and \( \triangle CDB \) are similar because \( \angle DAH = \angle BCD \), \( \angle ADH = \angle CDB \).

Therefore, \( \overline{AD}/\overline{CD} = \overline{HD}/\overline{DB}, \) i.e \( \overline{AD} \cdot \overline{DB} = \overline{HD} \cdot \overline{CD} \). Thus \( [\overline{ABE}]^2 = [\overline{ABC}][\overline{ABH}] \).

5. We have \( \frac{1}{a} + \frac{2}{b} + \frac{3}{c} = 1 \) and \( 1 \leq a \leq b \leq c \). Therefore \( 1/a \geq 1/b \geq 1/c \) and
\[
1 = \frac{1}{a} + \frac{2}{b} + \frac{3}{c} \leq \frac{1}{a} + \frac{2}{a} + \frac{3}{a} = \frac{6}{a} \Rightarrow a \leq 6.
\]
Since \( a \) is a prime and \( 1 \leq a \leq 6 \), the possible values of \( a \) are 2, 3 and 5.

**Case 1:** \( a = 2 \).
\[
a = 2 \Rightarrow \frac{2}{b} + \frac{3}{c} = \frac{1}{2} \text{. Now } \frac{2}{b} < \frac{1}{2} \Rightarrow b \geq 5 \text{ and } 1/b \geq 1/c \Rightarrow \frac{1}{2} = \frac{2}{b} + \frac{3}{c} \leq \frac{5}{b} \Rightarrow b \leq 10.
\]
Hence \( 5 \leq b \leq 10 \). Substituting the possible values of \( b \) in the equation \( \frac{2}{b} + \frac{3}{c} = \frac{1}{2} \) we obtain \( (b, c) = (5, 30), (6, 18), (7, 14), (8, 12), (10, 10) \) as the admissible pairs. Therefore the solutions in this case are \( (a, b, c) = (2, 5, 30), (2, 6, 18), (2, 7, 14), (2, 8, 12), (2, 10, 10), (3, 4, 18), (3, 6, 9) \).

**Case 2:** \( a = 3 \).
Emulating the method outlined in the analysis of **Case 1** we find that \( 4 \leq b \leq 7 \) and the solutions in this case are \( (a, b, c) = (3, 4, 18), (3, 6, 9) \).

**Case 3:** \( a = 5 \).
There is no solution in this case.
In summary, the solutions are \( (a, b, c) = (2, 5, 30), (2, 6, 18), (2, 7, 14), (2, 8, 12), (2, 10, 10), (3, 4, 18), (3, 6, 9) \).
6. WSUM = 70656. Partition the set of all subsets of $S$ into two sets $T_1$ consisting of those subsets which contain the element 1 and $T_2$ those which do not contain 1. For every subset $S_2$ belonging to $T_2$, there is a unique subset $S_1$ belonging to $T_1$ which is $\{1\} \cup S_2$. Consider any element $a \geq 2$ of $S$. Let $S_2$ be such that the element $a$ occupies an even numbered position in it. It will occupy an odd numbered position in $S_1$. The total contribution to WSUM by $a$ from both these subsets is $5a$. The same is true if $a$ occupies an odd numbered position in $S_2$. Therefore, the total contribution of the element $a$ to WSUM is $5a$ multiplied by the number of subsets of $T_2$ that contain $a$. The number of these subsets is $2^{8-1}$. The contribution of the element 1 to WSUM is clearly $3 \times 2^{9}$. Therefore the sum of all WSUMs is $T = 3 \times 2^{9} + 5 \times 2^{8} \times \sum_{j=2}^{10} j = 276 \times 2^{8} = 70656$.

Aliter: The total contribution of the element $a \geq 2$ over all subsets is $t = \sum_{a=2}^{A} (a-1) \left(2 + \xi(j) \right) 2^{10-a}$. Here $\xi(j) = 0$ if $j$ is odd and $\xi(j) = 1$ if $j$ is even. $(a-1)$ is the number of $j$-element subsets of $\{1,2,\ldots,a-1\}$ and $2^{10-a}$ is the total number of subsets of $\{a+1,a+2,\ldots,10\}$. Simplifying, we get $t = 2.2^{a} + 2^{8}a = 5.2^{8}a$.

7. $O$, $E$, and $X$ are collinear. Join $O$ with $A$, $B$ and $C$. Triangles $OCX$ and $CEX$ are similar.

\[ \frac{XC}{XO} = \frac{XE}{XC}, \text{ i.e } XC^2 = XO \cdot XE. \] But $XC^2 = XB \cdot XA$. Hence $XB \cdot XA = XE \cdot XO$ implying $B, A, O, E$ are concyclic. Therefore $\angle OAB = 180^\circ - \angle OEB = 40^\circ$. So, $\angle AOB = 180^\circ - 2 \angle OAB = 100^\circ$.

8. Put $x = 2a$, $y = 2b$ and $z = 2c$. The problem reduces to showing

\[
\frac{1}{(a-1)(b-1)(c-1)} + \frac{8}{(a+1)(b+1)(c+1)} \leq \frac{1}{4}
\]

subject to $1/a + 1/b + 1/c = 1$. Observe that $a > 1, b > 1, c > 1$ and $a-1 = a \left(\frac{1}{b} + \frac{1}{c}\right) \geq \frac{2a}{\sqrt{bc}}$ (by A.M-G.M inequality). Similarly we get $b-1 \geq \frac{2b}{\sqrt{ca}}$ and $c-1 \geq \frac{2c}{\sqrt{ab}}$. Multiplying these and taking the reciprocal we obtain

\[
\frac{1}{(a-1)(b-1)(c-1)} \leq \frac{1}{8} \ldots (1)
\]

Next observe that $\frac{a + 1}{a - 1} = 1 + \frac{2}{a - 1} \geq \frac{2\sqrt{2}}{\sqrt{a - 1}}$ whence $a + 1 \geq 2\sqrt{2(a - 1)}$. Similarly we obtain $b + 1 \geq 2\sqrt{2(b - 1)}$ and $c + 1 \geq 2\sqrt{2(c - 1)}$. Multiplying these yields

\[
(a + 1)(b + 1)(c + 1) \geq 16\sqrt{2(a - 1)(b - 1)(c - 1)} \geq 16\sqrt{2.8} = 64.
\]
Therefore
\[
\frac{8}{(a + 1)(b + 1)(c + 1)} \leq \frac{1}{8}. \quad \ldots \text{(II)}
\]

By adding (I) and (II) we get
\[
\frac{1}{(a - 1)(b - 1)(c - 1)} + \frac{8}{(a + 1)(b + 1)(c + 1)} \leq \frac{1}{4}.
\]

**Comments:**

We present another method which many students had adopted. By the A.M-G.M-H.M inequality,
\[
\frac{a + b + c}{3} \geq \sqrt[3]{abc} \geq \frac{3}{1/a + 1/b + 1/c} = 3.
\]

Thus \(a + b + c \geq 9\) and \(abc \geq 27\). Using these two inequalities we obtain
\[
(a - 1)(b - 1)(c - 1) = abc - (ab + bc + ca) + (a + b + c) - 1 \geq 8,
(a+1)(b+1)(c+1) = abc+(ab+bc+ca)+(a+b+c)+1 = 2abc+(a+b+c)+1 \geq 2(27)+9+1 = 64.
\]

From these two inequalities we get
\[
\frac{1}{(a - 1)(b - 1)(c - 1)} + \frac{8}{(a + 1)(b + 1)(c + 1)} \leq \frac{1}{4}.
\]