1. Let \( ABC \) be an acute-angled triangle and let \( D, E, F \) be the feet of perpendiculars from \( A, B, C \) respectively to \( BC, CA, AB \). Let the perpendiculars from \( F \) to \( CB, CA, AD, BE \) meet them in \( P, Q, M, N \) respectively. Prove that \( P, Q, M, N \) are collinear.

**Solution:** Observe that \( C, Q, F, P \) are concyclic. Hence
\[
\angle CQP = \angle CFP = 90^\circ - \angle FCP = \angle B.
\]
Similarly the concyclicity of \( F, M, Q, A \) gives
\[
\angle AQN = 90^\circ + \angle QFM = 90^\circ + \angle FAM = 90^\circ + 90^\circ - \angle B = 180^\circ - \angle B.
\]
Thus we obtain \( \angle CQP + \angle AQN = 180^\circ \). It follows that \( Q, N, P \) lie on the same line.

![Diagram](https://via.placeholder.com/150)

We can similarly prove that \( \angle CPQ + \angle BPM = 180^\circ \). This implies that \( P, M, Q \) are collinear. Thus \( M, N \) both lie on the line joining \( P \) and \( Q \).

2. Find the least possible value of \( a + b \), where \( a, b \) are positive integers such that 11 divides \( a + 13b \) and 13 divides \( a + 11b \).

**Solution:** Since 13 divides \( a + 11b \), we see that 13 divides \( a - 2b \) and hence it also divides \( 6a - 12b \). This in turn implies that \( 13|(6a + b) \). Similarly \( 11|(a + 13b) \Rightarrow 11|(a + 2b) \Rightarrow 11|(6a + 12b) \Rightarrow 11|(6a + b) \). Since \( \gcd(11, 13) = 1 \), we conclude that \( 143|(6a + b) \). Thus we may write \( 6a + b = 143k \) for some natural number \( k \). Hence
\[
6a + 6b = 143k + 5b = 144k + 6b - (k + b).
\]
This shows that 6 divides \( k + b \) and hence \( k + b \geq 6 \). We therefore obtain
\[
6(a + b) = 143k + 5b = 138k + 5(k + b) \geq 138 + 5 \times 6 = 168.
\]
It follows that \( a + b \geq 28 \). Taking \( a = 23 \) and \( b = 5 \), we see that the conditions of the problem are satisfied. Thus the minimum value of \( a + b \) is 28.

3. If \( a, b, c \) are three positive real numbers, prove that

\[
\frac{a^2 + 1}{b + c} + \frac{b^2 + 1}{c + a} + \frac{c^2 + 1}{a + b} \geq 3.
\]

**Solution:** We use the trivial inequalities \( a^2 + 1 \geq 2a \), \( b^2 + 1 \geq 2b \) and \( c^2 + 1 \geq 2c \). Hence we obtain
\[
\frac{a^2 + 1}{b + c} + \frac{b^2 + 1}{c + a} + \frac{c^2 + 1}{a + b} \geq \frac{2a}{b + c} + \frac{2b}{c + a} + \frac{2c}{a + b}.
\]
Adding 6 both sides, this is equivalent to

\[
(2a + 2b + 2c) \left( \frac{1}{b + c} + \frac{1}{c + a} + \frac{1}{a + b} \right) \geq 9.
\]

Taking \( x = b + c, y = c + a, z = a + b \), this is equivalent to

\[
(x + y + z) \left( \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) \geq 9.
\]

This is a consequence of AM-GM inequality.

**Alternately:** The substitutions \( b + c = x, c + a = y, a + b = z \) leads to

\[
\sum \frac{2a}{b+c} = \sum \frac{y + z - x}{x} = \sum \left( \frac{x}{y} + \frac{y}{x} \right) - 3 \geq 6 - 3 = 3.
\]

4. A \( 6 \times 6 \) square is dissected into 9 rectangles by lines parallel to its sides such that all these rectangles have integer sides. Prove that there are always two congruent rectangles.

**Solution:** Consider the dissection of the given \( 6 \times 6 \) square in to non-congruent rectangles with least possible areas. The only rectangle with area 1 is an \( 1 \times 1 \) rectangle. Similarly, we get \( 1 \times 2, 1 \times 3 \) rectangles for areas \( 2, 3 \) units. In the case of 4 units we may have either a \( 1 \times 4 \) rectangle or a \( 2 \times 2 \) square. Similarly, there can be a \( 1 \times 5 \) rectangle for area 5 units and \( 1 \times 6 \) or \( 2 \times 3 \) rectangle for 6 units. Any rectangle with area 7 units must be \( 1 \times 7 \) rectangle, which is not possible since the largest side could be 6 units. And any rectangle with area 8 units must be a \( 2 \times 4 \) rectangle. If there is any dissection of the given \( 6 \times 6 \) square in to 9 non-congruent rectangles with areas \( a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5 \leq a_6 \leq a_7 \leq a_8 \leq a_9 \), then we observe that

\[
a_1 \geq 1, \ a_2 \geq 2, \ a_3 \geq 3, \ a_4 \geq 4, \ a_5 \geq 4, \ a_6 \geq 5, \ a_7 \geq 6, \ a_8 \geq 6, \ a_9 \geq 8,
\]

and hence the total area of all the rectangles is

\[
a_1 + a_2 + \cdots + a_9 \geq 1 + 2 + 3 + 4 + 4 + 5 + 6 + 6 + 8 = 39 > 36,
\]

which is the area of the given square. Hence if a \( 6 \times 6 \) square is dissected in to 9 rectangles as stipulated in the problem, there must be two congruent rectangles.

5. Let \( ABCD \) be a quadrilateral in which \( AB \) is parallel to \( CD \) and perpendicular to \( AD \); \( AB = 3CD \); and the area of the quadrilateral is 4. If a circle can be drawn touching all the sides of the quadrilateral, find its radius.

**Solution:** Let \( P, Q, R, S \) be the points of contact of in-circle with the sides \( AB, BC, CD, DA \) respectively. Since \( AD \) is perpendicular to \( AB \) and \( AB \) is parallel to \( DC \), we see that \( AP = AS = SD = DR = r \), the radius of the inscribed circle. Let \( BP = BQ = y \) and \( CQ = CR = x \). Using \( AB = 3CD \), we get \( r + y = 3(r + x) \).
Since the area of $ABCD$ is 4, we also get

$$4 = \frac{1}{2} AD(AB + CD) = \frac{1}{2}(2r)(4(r + x)).$$

Thus we obtain $r(r + x) = 1$. Using Pythagoras theorem, we obtain $BC^2 = BK^2 + CK^2$. However $BC = y + x$, $BK = y - x$ and $CK = 2r$. Substituting these and simplifying, we get $xy = r^2$. But $r + y = 3(r + x)$ gives $y = 2r + 3x$. Thus $r^2 = x(2r + 3x)$ and this simplifies to $(r - 3x)(r + x) = 0$. We conclude that $r = 3x$. Now the relation $r(r + x) = 1$ implies that $4r^2 = 3$, giving $r = \sqrt{3}/2$.

6. Prove that there are infinitely many positive integers $n$ such that $n(n+1)$ can be expressed as a sum of two positive squares in at least two different ways. (Here $a^2 + b^2$ and $b^2 + a^2$ are considered as the same representation.)

**Solution:** Let $Q = n(n+1)$. It is convenient to choose $n = m^2$, for then $Q$ is already a sum of two squares: $Q = m^2(m^2 + 1) = (m^2)^2 + m^2$. If further $m^2$ itself is a sum of two squares, say $m^2 = p^2 + q^2$, then

$$Q = (p^2 + q^2)(m^2 + 1) = (pm + q)^2 + (p - qm)^2.$$  

Note that the two representations for $Q$ are distinct. Thus, for example, we may take $m = 5k, p = 3k, q = 4k$, where $k$ varies over natural numbers. In this case $n = m^2 = 25k^2$, and

$$Q = (25k^2)^2 + (5k)^2 = (15k^2 + 4k)^2 + (20k^2 - 3k)^2.$$  

As we vary $k$ over natural numbers, we get infinitely many numbers of the form $n(n+1)$ each of which can be expressed as a sum of two squares in two distinct ways.

7. Let $X$ be the set of all positive integers greater than or equal to 8 and let $f : X \to X$ be a function such that $f(x + y) = f(xy)$ for all $x \geq 4, y \geq 4$. If $f(8) = 9$, determine $f(9)$.

**Solution:** We observe that

$$f(9) = f(4 + 5) = f(4 \cdot 5) = f(20) = f(16 + 4) = f(16 \cdot 4) = f(64) = f(8 \cdot 8) = f(8 + 8) = f(16) = f(4 \cdot 4) = f(4 + 4) = f(8).$$

Hence if $f(8) = 9$, then $f(9) = 9$. (This is one string. There may be other different ways of approaching $f(8)$ from $f(9)$. The important thing to be observed is the fact that the rule $f(x + y) = f(xy)$ applies only when $x$ and $y$ are at least 4. One may get strings using numbers $x$ and $y$ which are smaller than 4, but that is not valid. For example

$$f(9) = f(3 \cdot 3) = f(3 + 3) = f(6) = f(4 + 2) = f(4 \cdot 2) = f(8),$$

is not a valid string.)