1. Let $ABC$ be a right-angled triangle with $\angle B = 90^\circ$. Let $BD$ be the altitude from $B$ on to $AC$. Let $P$, $Q$ and $I$ be the incentres of triangles $ABD$, $CBD$ and $ABC$ respectively. Show that the circumcentre of the triangle $PIQ$ lies on the hypotenuse $AC$.

**Solution:** We begin with the following lemma:

**Lemma:** Let $XYZ$ be a triangle with $\angle XYZ = 90 + \alpha$. Construct an isosceles triangle $XEZ$, externally on the side $XZ$, with base angle $\alpha$. Then $E$ is the circumcentre of $\triangle XYZ$.

**Proof of the Lemma:** Draw $ED \perp XZ$. Then $DE$ is the perpendicular bisector of $XZ$. We also observe that $\angle XED = \angle ZED = 90 - \alpha$. Observe that $E$ is on the perpendicular bisector of $XZ$. Construct the circumcircle of $XYZ$. Draw perpendicular bisector of $XY$ and let it meet $DE$ in $F$. Then $F$ is the circumcentre of $\triangle XYZ$. Join $XF$. Then $\angle XFD = 90 - \alpha$. But we know that $\angle XED = 90 - \alpha$. Hence $E = F$.

Let $r_1$, $r_2$ and $r$ be the inradii of the triangles $ABD$, $CBD$ and $ABC$ respectively. Join $PD$ and $DQ$. Observe that $\angle PDQ = 90^\circ$. Hence

$$PQ^2 = PD^2 + DQ^2 = 2r_1^2 + 2r_2^2.$$

Let $s_1 = (AB + BD + DA)/2$. Observe that $BD = ca/b$ and $AD = \sqrt{AB^2 - BD^2} = \sqrt{c^2 - \left(\frac{ca}{b}\right)^2} = c^2/b$. This gives $s_1 = cs/b$. But $r_1 = s_1 - c = (c/b)(s - b) = cr/b$. Similarly, $r_2 = ar/b$. Hence

$$PQ^2 = 2r^2 \left(\frac{c^2 + a^2}{b^2}\right) = 2r^2.$$

Consider $\triangle PIQ$. Observe that $\angle PIQ = 90 + (B/2) = 135$. Hence $PQ$ subtends $90^\circ$ on the circumference of the circumcircle of $\triangle PIQ$. But we have seen that $\angle PDQ = 90^\circ$. Now construct a circle with $PQ$ as diameter. Let it cut $AC$ again in $K$. It follows that $\angle PKQ = 90^\circ$ and the points $P, D, K, Q$ are concyclic. We also notice $\angle KPQ = \angle KDPQ = 45^\circ$ and $\angle PQK = \angle PDA = 45^\circ$. 
Thus $PKQ$ is an isosceles right-angled triangle with $KP = KQ$. Therefore $KP^2 + KQ^2 = PQ^2 = 2r^2$ and hence $KP = KQ = r$.

Now $\angle PIQ = 90 + 45$ and $\angle PKQ = 2 \times 45^\circ = 90^\circ$ with $KP = KQ = r$. Hence $K$ is the circumcentre of $\triangle PIQ$.

(Incidentally, This also shows that $KI = r$ and hence $K$ is the point of contact of the incircle of $\triangle ABC$ with $AC$.)

**Solution 2:** Here we use computation to prove that the point of contact $K$ of the incircle with $AC$ is the circumcentre of $\triangle PIQ$. We show that $KP = KQ = r$. Let $r_1$ and $r_2$ be the inradii of triangles $ABD$ and $CBD$ respectively. Draw $PL \perp AC$ and $QM \perp AC$. If $s_1$ is the semiperimeter of $\triangle ABD$, then $AL = s_1 - BD$.

But

$$s_1 = \frac{AB + BD + DA}{2}, \quad BD = \frac{ca}{b}, \quad AD = \frac{c^2}{b}$$

Hence $s_1 = cs/b$. This gives $r_1 = s_1 - c = cr/b$, $AL = s_1 - BD = c(s-a)/b$.

Hence $KL = AK - AL = (s - a) - \frac{c(s-a)}{b} = \frac{(b-c)(s-a)}{b}$. We observe that

$$2r^2 = \frac{(c + a - b)^2}{2} = \frac{c^2 + a^2 + b^2 - 2bc - 2ab + 2ca}{2} = \frac{(b^2 - ba - bc + ac)}{(b-c)(b-a)}.$$

This gives

$$(s-a)(b-c) = (s-b + b-a)(b-c) = r(b-c) + (b-a)(b-c)$$

$$= r(b-c) + 2r^2 = r(b-c + c + a - b) = ra.$$

Thus $KL = ra/b$. Finally,

$$KP^2 = KL^2 + LP^2 = \frac{r^2a^2}{b^2} + \frac{r^2 + c^2}{b^2} = r^2.$$ 

Thus $KP = r$. Similarly, $KQ = r$. This gives $KP = KI = KQ = r$ and therefore $K$ is the circumcentre of $\triangle KIQ$.

(Incidentally, this also shows that $KL = ca/b = r_2$ and $KM = r_1$.)

2. For any natural number $n > 1$, write the infinite decimal expansion of $1/n$ (for example, we write $1/2 = 0.4\bar{9}$ as its infinite decimal expansion, not $0.5$). Determine the length of the non-periodic part of the (infinite) decimal expansion of $1/n$.

**Solution:** For any prime $p$, let $\nu_p(n)$ be the maximum power of $p$ dividing $n$; i.e $p^{\nu_p(n)}$ divides $n$ but not higher power. Let $r$ be the
length of the non-periodic part of the infinite decimal expansion of $1/n$. 

Write \[ \frac{1}{n} = 0.a_1a_2 \cdots a_r\overline{b_1b_2 \cdots b_s}. \]

We show that $r = \max(\nu_2(n), \nu_5(n))$.

Let $a$ and $b$ be the numbers $a_1a_2 \cdots a_r$ and $b = b_1b_2 \cdots b_s$ respectively. (Here $a_1$ and $b_1$ can be both 0.) Then

\[
\frac{1}{n} = \frac{1}{10^r} \left( a + \sum_{k \geq 1} \frac{b}{(10^s)^k} \right) = \frac{1}{10^r} \left( a + \frac{b}{10^s - 1} \right).
\]

Thus we get $10^r(10^s - 1) = n((10^s - 1)a + b)$. It shows that $r \geq \max(\nu_2(n), \nu_5(n))$. Suppose $r > \max(\nu_2(n), \nu_5(n))$. Then 10 divides $b - a$. Hence the last digits of $a$ and $b$ are equal: $a_r = b_s$. This means

\[
\frac{1}{n} = 0.a_1a_2 \cdots a_{r-1}\overline{b_s b_1 b_2 \cdots b_{s-1}}.
\]

This contradicts the definition of $r$. Therefore $r = \max(\nu_2(n), \nu_5(n))$.

3. Find all real functions $f$ from $\mathbb{R} \rightarrow \mathbb{R}$ satisfying the relation

\[ f(x^2 + yf(x)) = xf(x+y). \]

**Solution:** Put $x = 0$ and we get $f(yf(0)) = 0$. If $f(0) \neq 0$, then $yf(0)$ takes all real values when $y$ varies over real line. We get $f(x) \equiv 0$. Suppose $f(0) = 0$. Taking $y = -x$, we get $f(x^2 - xf(x)) = 0$ for all real $x$.

Suppose there exists $x_0 \neq 0$ in $\mathbb{R}$ such that $f(x_0) = 0$. Putting $x = x_0$ in the given relation we get

\[ f(x_0^2) = x_0 f(x_0 + y), \]

for all $y \in \mathbb{R}$. Now the left side is a constant and hence it follows that $f$ is a constant function. But the only constant function which satisfies the equation is identically zero function, which is already obtained. Hence we may consider the case where $f(x) \neq 0$ for all $x \neq 0$.

Since $f(x^2 - xf(x)) = 0$, we conclude that $x^2 - xf(x) = 0$ for all $x \neq 0$. This implies that $f(x) = x$ for all $x \neq 0$. Since $f(0) = 0$, we conclude that $f(x) = x$ for all $x \in \mathbb{R}$.

Thus we have two functions: $f(x) \equiv 0$ and $f(x) = x$ for all $x \in \mathbb{R}$.

4. There are four basket-ball players $A, B, C, D$. Initially, the ball is with $A$. The ball is always passed from one person to a different person. In how many ways can the ball come back to $A$ after seven passes? (For example $A \rightarrow C \rightarrow B \rightarrow D \rightarrow A \rightarrow B \rightarrow C \rightarrow A$ and
$A \to D \to A \to D \to C \to A \to B \to A$ are two ways in which the ball can come back to $A$ after seven passes.

**Solution:** Let $x_n$ be the number of ways in which $A$ can get back the ball after $n$ passes. Let $y_n$ be the number of ways in which the ball goes back to a fixed person other than $A$ after $n$ passes. Then

$$x_n = 3y_{n-1},$$

and

$$y_n = x_{n-1} + 2y_{n-1}.$$  

We also have $x_1 = 0$, $x_2 = 3$, $y_1 = 1$ and $y_2 = 2$.

Eliminating $y_n$ and $y_{n-1}$, we get (Alternate solution)

$$x_{n+1} = 3x_n - 1 + 2x_n - 1,$$

we get

$$x_{n+1} = 3x_n - 1 + 2x_n = 3x_n + 2x_n = 3^n - 1 + 2^{n-1}.$$  

Thus

$$x_3 = 3x_2 + 2x_3 = 3 \times 3 = 9;$$  

$$x_4 = 3x_3 + 2x_4 = (3 \times 3) + (2 \times 6) = 9 + 12 = 21;$$  

$$x_5 = 3x_4 + 2x_5 = (3 \times 21) + (2 \times 6) = 63 + 120 = 183;$$  

$$x_6 = 3x_5 + 2x_6 = (3 \times 183) + (2 \times 21) = 549 + 42 = 591;$$  

$$x_7 = 3x_6 + 2x_7 = (3 \times 549) + (2 \times 183) = 1647 + 366 = 2013.$$

**Alternate solution:** Since the ball goes back to one of the other 3 persons, we have

$$x_n + 3y_n = 3^n,$$

since there are $3^n$ ways of passing the ball in $n$ passes. Using $x_n = 3y_{n-1}$, we obtain

$$x_{n-1} + x_n = 3^{n-1},$$

with $x_1 = 0$. Thus

$$x_7 = 3^6 - x_6 = 3^6 - 3^5 + x_5 = 3^6 - 3^5 + 3^4 - x_4 = 3^6 - 3^5 + 3^4 - 3^3 + x_3$$

$$= 3^6 - 3^5 + 3^4 - 3^3 + 3^2 - x_2 = 3^6 - 3^5 + 3^4 - 3^3 + 3^2 - 3$$

$$= (2 \times 3^5) + (2 \times 3^3) + (2 \times 3) = 486 + 54 + 6 = 546.$$

5. Let $ABCD$ be a convex quadrilateral. Let the diagonals $AC$ and $BD$ intersect in $P$. Let $PE$, $PF$, $PG$ and $PH$ be the altitudes from $P$ on to the sides $AB$, $BC$, $CD$ and $DA$ respectively. Show that $ABCD$ has an incircle if and only if

$$\frac{1}{PE} + \frac{1}{PG} = \frac{1}{PF} + \frac{1}{PH}.$$  

**Solution:** Let $AP = p$, $BP = q$, $CP = r$, $DP = s$; $AB = a$, $BC = b$, $CD = c$ and $DA = d$. Let $\angle APB = \angle CPD = \theta$. Then $\angle BPC = \angle DPA = \pi - \theta$. Let us also write $PE = h_1$, $PF = h_2$, $PG = h_3$ and $PH = h_4.$
Observe that

\[ h_1 a = pq \sin \theta, \quad h_2 b = qr \sin \theta, \quad h_3 c = rs \sin \theta, \quad h_4 d = sp \sin \theta. \]

Hence

\[ \frac{1}{h_1} + \frac{1}{h_3} = \frac{1}{h_2} + \frac{1}{h_4}. \]

is equivalent to

\[ \frac{a}{pq} + \frac{c}{rs} = \frac{b}{qr} + \frac{d}{sp}. \]

This is the same as

\[ ars + cpq = bsp + dqr. \]

Thus we have to prove that \( a + c = b + d \) if and only if \( ars + cpq = bsp + dqr \).
Now we can write \( a + c = b + d \) as

\[ a^2 + c^2 + 2ac = b^2 + d^2 + 2bd. \]

But we know that

\[ a^2 = p^2 + q^2 - 2pq \cos \theta, \quad c^2 = r^2 + s^2 - 2rs \cos \theta \]
\[ b^2 = q^2 + r^2 + 2qr \cos \theta, \quad d^2 = p^2 + s^2 + 2ps \cos \theta. \]

Hence \( a + c = b + d \) is equivalent to

\[ -pq \cos \theta + -rs \cos \theta + ac = ps \cos \theta + qr \cos \theta + bd. \]

Similarly, by squaring \( ars + cpq = bsp + dqr \) we can show that it is equivalent to

\[ -pq \cos \theta + -rs \cos \theta + ac = ps \cos \theta + qr \cos \theta + bd. \]

We conclude that \( a + c = b + d \) is equivalent to \( cpq + ars = bps + dqr \).
Hence \( ABCD \) has an in circle if and only if

\[ \frac{1}{h_1} + \frac{1}{h_3} = \frac{1}{h_2} + \frac{1}{h_4}. \]
6. From a set of 11 square integers, show that one can choose 6 numbers $a^2, b^2, c^2, d^2, e^2, f^2$ such that

$$a^2 + b^2 + c^2 \equiv d^2 + e^2 + f^2 \pmod{12}.$$ 

**Solution:** The first observation is that we can find 5 pairs of squares such that the two numbers in a pair have the same parity. We can see this as follows:

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<th>Odd numbers</th>
<th>Even numbers</th>
<th>Odd pairs</th>
<th>Even pairs</th>
<th>Total pairs</th>
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<td>5</td>
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</tbody>
</table>

Let us take such 5 pairs: say $(x_1^2, y_1^2), (x_2^2, y_2^2), \ldots, (x_5^2, y_5^2)$. Then $x_j^2 - y_j^2$ is divisible by 4 for $1 \leq j \leq 5$. Let $r_j$ be the remainder when $x_j^2 - y_j^2$ is divisible by 3, $1 \leq j \leq 3$. We have 5 remainders $r_1, r_2, r_3, r_4, r_5$. But these can be 0, 1 or 2. Hence either one of the remainders occur 3 times or each of the remainders occur once. If, for example $r_1 = r_2 = r_3$, then 3 divides $r_1 + r_2 + r_3$; if $r_1 = 0, r_2 = 1$ and $r_3 = 2$, then again 3 divides $r_1 + r_2 + r_3$. Thus we can always find three remainders whose sum is divisible by 3. This means we can find 3 pairs, say, $(x_1^2, y_1^2), (x_2^2, y_2^2), (x_3^2, y_3^2)$ such that 3 divides $(x_1^2 - y_1^2) + (x_2^2 - y_2^2) + (x_3^2 - y_3^2)$. Since each difference is divisible by 4, we conclude that we can find 6 numbers $a^2, b^2, c^2, d^2, e^2, f^2$ such that

$$a^2 + b^2 + c^2 \equiv d^2 + e^2 + f^2 \pmod{12}.$$