1. Let $ABC$ be a triangle with circum-circle $\Gamma$. Let $M$ be a point in the interior of triangle $ABC$ which is also on the bisector of $\angle A$. Let $AM$, $BM$, $CM$ meet $\Gamma$ in $A_1$, $B_1$, $C_1$ respectively. Suppose $P$ is the point of intersection of $A_1C_1$ with $AB$; and $Q$ is the point of intersection of $A_1B_1$ with $AC$. Prove that $PQ$ is parallel to $BC$.

**Solution:** Let $A = 2\alpha$. Then $\angle A_1AC = \angle BAA_1 = \alpha$. Thus

$$\angle A_1B_1C = \alpha = \angle BB_1A_1 = \angle A_1C_1C = \angle BC_1A_1.$$  

We also have $\angle B_1CQ = \angle AA_1B_1 = \beta$, say. It follows that triangles $MA_1B_1$ and $QCB_1$ are similar and hence

$$\frac{QC}{MA_1} = \frac{B_1C}{B_1A_1}.$$  

Similarly, triangles $ACM$ and $C_1A_1M$ are similar and we get

$$\frac{AC}{AM} = \frac{C_1A_1}{C_1M}.$$  

Using the point $P$, we get similar ratios:

$$\frac{PB}{MA_1} = \frac{C_1B}{A_1C_1}, \quad \frac{AB}{AM} = \frac{A_1B_1}{MB_1}.$$  

Thus,

$$\frac{QC}{PB} = \frac{A_1C_1 \cdot B_1C}{C_1B \cdot B_1A_1},$$

and

$$\frac{AC}{AB} = \frac{MB_1 \cdot C_1A_1}{A_1B_1 \cdot C_1M} = \frac{MB_1^2 \cdot C_1B}{C_1M^2 \cdot A_1B_1} = \frac{MB_1 \cdot C_1B \cdot QC}{C_1M^2 \cdot PB \cdot B_1C}.$$  

However, triangles $C_1BM$ and $B_1CM$ are similar, which gives

$$\frac{B_1C}{C_1B} = \frac{MB_1}{MC_1}.$$
2. Find all natural numbers \( n > 1 \) such that \( n^2 \) does not divide \((n - 2)!\).

**Solution:** Suppose \( n = pqr \), where \( p < q \) are primes and \( r > 1 \). Then \( p \geq 2 \), \( q \geq 3 \) and \( r \geq 2 \), not necessarily a prime. Thus we have

\[
\begin{align*}
n - 2 & \geq n - p = pqr - p \geq 5p > p, \\
n - 2 & \geq n - q = q(pr - 1) \geq 3q > q, \\
n - 2 & \geq n - pr = pr(q - 1) \geq 2pr > pr, \\
n - 2 & \geq n - qr = qr(p - 1) \geq qr.
\end{align*}
\]

Observe that \( p, q, pr, qr \) are all distinct. Hence their product divides \((n - 2)!\). Thus \( n^2 = p^2 q^2 r^2 \) divides \((n - 2)!\) in this case. We conclude that either \( n = pq \) where \( p, q \) are distinct primes or \( n = p^k \) for some prime \( p \).

**Case 1.** Suppose \( n = pq \) for some primes \( p, q \), where \( 2 < p < q \). Then \( p \geq 3 \) and \( q \geq 5 \).

In this case

\[
\begin{align*}
n - 2 & > n - p = p(q - 1) \geq 4p, \\
n - 2 & > n - q = q(p - 1) \geq 2q.
\end{align*}
\]

Thus \( p, q, 2p, 2q \) are all distinct numbers in the set \( \{1, 2, 3, \ldots, n - 2\} \). We see that \( n^2 = p^2 q^2 \) divides \((n - 2)!\). We conclude that \( n = 2q \) for some prime \( q \geq 3 \). Note that \( n - 2 = 2q - 2 < 2q \) in this case so that \( n^2 \) does not divide \((n - 2)!\).

**Case 2.** Suppose \( n = p^k \) for some prime \( p \). We observe that \( p, 2p, 3p, \ldots, (p^{k-1} - 1)p \) all lie in the set \( \{1, 2, 3, \ldots, n - 2\} \). If \( p^{k-1} - 1 \geq 2k \), then there are at least \( 2k \) multiples of \( p \) in the set \( \{1, 2, 3, \ldots, n - 2\} \). Hence \( n^2 = p^{2k} \) divides \((n - 2)!\). Thus \( p^{k-1} - 1 < 2k \).

If \( k \geq 5 \), then \( p^{k-1} - 1 \geq 2k - 1 - 1 \geq 2k \), which may be proved by an easy induction. Hence \( k \leq 4 \). If \( k = 1 \), we get \( n = p \), a prime. If \( k = 2 \), then \( p - 1 < 4 \) so that \( p = 2 \) or \( 3 \); we get \( n = 2^2 = 4 \) or \( n = 3^2 = 9 \). For \( k = 3 \), we have \( p^2 - 1 < 6 \) giving \( p = 2 \); \( n = 2^3 = 8 \) in this case. Finally, \( k = 4 \) gives \( p^3 - 1 < 8 \). Again \( p = 2 \) and \( n = 2^4 = 16 \). However \( n^2 = 2^8 \) divides \( 14! \) and hence is not a solution.

Thus \( n = p, 2p \) for some prime \( p \) or \( n = 8, 9 \). It is easy to verify that these satisfy the conditions of the problem.

3. Find all non-zero real numbers \( x, y, z \) which satisfy the system of equations:

\[
\begin{align*}
(x^2 + xy + y^2)(y^2 + yz + z^2)(z^2 + zx + x^2) &= xyz, \\
(x^4 + x^2 y^2 + y^4)(y^4 + y^2 z^2 + z^4)(z^4 + z^2 x^2 + x^4) &= x^3 y^3 z^3.
\end{align*}
\]

**Solution:** Since \( xyz \neq 0 \), We can divide the second relation by the first. Observe that

\[
x^4 + x^2 y^2 + y^4 = (x^2 + xy + y^2)(x^2 - xy + y^2),
\]

holds for any \( x, y \). Thus we get

\[
(x^2 - xy + y^2)(y^2 - yz + z^2)(z^2 - zx + x^2) = x^2 y^2 z^2.
\]
However, for any real numbers \( x, y \), we have

\[
x^2 - xy + y^2 \geq |xy|.
\]

Since \( x^2y^2z^2 = |xy| |yz| |zx| \), we get

\[
|xy| |yz| |zx| = (x^2 - xy + y^2)(y^2 - yz + z^2)(z^2 - zx + x^2) \geq |xy| |yz| |zx|.
\]

This is possible only if

\[
x^2 - xy + y^2 = |xy|, \quad y^2 - yz + z^2 = |yz|, \quad z^2 - zx + x^2 = |zx|,
\]

hold simultaneously. However \( |xy| = \pm xy \). If \( x^2 - xy + y^2 = -xy \), then \( x^2 + y^2 = 0 \) giving \( x = y = 0 \). Since we are looking for nonzero \( x, y, z \), we conclude that \( x^2 - xy + y^2 = xy \) which is same as \( x = y \). Using the other two relations, we also get \( y = z \) and \( z = x \). The first equation now gives \( 27x^6 = x^3 \). This gives \( x^3 = 1/27 \) (since \( x \neq 0 \)), or \( x = 1/3 \). We thus have \( x = y = z = 1/3 \). These also satisfy the second relation, as may be verified.

4. How many 6-tuples \((a_1, a_2, a_3, a_4, a_5, a_6)\) are there such that each of \( a_1, a_2, a_3, a_4, a_5, a_6 \) is from the set \( \{1, 2, 3, 4\} \) and the six expressions

\[
a_j^2 - a_ja_{j+1} + a_{j+1}^2
\]

for \( j = 1, 2, 3, 4, 5, 6 \) (where \( a_7 \) is to be taken as \( a_1 \)) are all equal to one another?

**Solution:** Without loss of generality, we may assume that \( a_1 \) is the largest among \( a_1, a_2, a_3, a_4, a_5, a_6 \). Consider the relation

\[
a_1^2 - a_1a_2 + a_2^2 = a_2^2 - a_2a_3 + a_3^2.
\]

This leads to

\[
(a_1 - a_3)(a_1 + a_3 - a_2) = 0.
\]

Observe that \( a_1 \geq a_2 \) and \( a_3 > 0 \) together imply that the second factor on the left side is positive. Thus \( a_1 = a_3 = \max\{a_1, a_2, a_3, a_4, a_5, a_6\} \). Using this and the relation

\[
a_3^2 - a_3a_4 + a_4^2 = a_4^2 - a_4a_5 + a_5^2,
\]

we conclude that \( a_3 = a_5 \) as above. Thus we have

\[
a_1 = a_3 = a_5 = \max\{a_1, a_2, a_3, a_4, a_5, a_6\}.
\]

Let us consider the other relations. Using

\[
a_2^2 - a_2a_3 + a_3^2 = a_3^2 - a_3a_4 + a_4^2,
\]

we get \( a_2 = a_4 \) or \( a_2 + a_4 = a_3 = a_1 \). Similarly, two more relations give either \( a_4 = a_6 \) or \( a_4 + a_6 = a_5 = a_1 \); and either \( a_6 = a_2 \) or \( a_6 + a_2 = a_1 \). Let us give values to \( a_1 \) and count the number of six-tuples in each case.

(A) Suppose \( a_1 = 1 \). In this case all \( a_j \)’s are equal and we get only one six-tuple \((1, 1, 1, 1, 1, 1)\).

(B) If \( a_1 = 2 \), we have \( a_3 = a_5 = 2 \). We observe that \( a_2 = a_4 = a_6 = 1 \) or \( a_2 = a_4 = a_6 = 2 \). We get two more six-tuples: \((2, 1, 2, 1, 2, 1)\), \((2, 2, 2, 2, 2, 2)\).

(C) Taking \( a_1 = 3 \), we see that \( a_3 = a_5 = 3 \). In this case we get nine possibilities for \((a_2, a_4, a_6)\):

\[
(1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 1, 2), (1, 2, 1), (2, 1, 1), (1, 2, 2), (2, 1, 2), (2, 2, 1).
\]
(D) In the case \(a_1 = 4\), we have \(a_3 = a_5 = 4\) and

\[
(a_2, a_4, a_6) = (2, 2, 2), (4, 4, 4), (1, 1, 1), (3, 3, 3),
(1, 1, 3), (1, 3, 1), (3, 1, 1), (3, 1, 3), (3, 3, 1).
\]

Thus we get \(1 + 2 + 9 + 10 = 22\) solutions. Since \((a_1, a_3, a_5)\) and \((a_2, a_4, a_6)\) may be interchanged, we get 22 more six-tuples. However there are 4 common among these, namely, \((1, 1, 1, 1, 1, 1)\), \((2, 2, 2, 2, 2, 2)\), \((3, 3, 3, 3, 3)\) and \((4, 4, 4, 4, 4)\). Hence the total number of six-tuples is \(22 + 22 - 4 = 40\).

5. Let \(ABC\) be an acute-angled triangle with altitude \(AK\). Let \(H\) be its ortho-centre and \(O\) be its circum-centre. Suppose \(KOH\) is an acute-angled triangle and \(P\) its circum-centre. Let \(Q\) be the reflection of \(P\) in the line \(HO\). Show that \(Q\) lies on the line joining the mid-points of \(AB\) and \(AC\).

**Solution:** Let \(D\) be the mid-point of \(BC\); \(M\) that of \(HK\); and \(T\) that of \(OH\). Then \(PM\) is perpendicular to \(HK\) and \(PT\) is perpendicular to \(OH\). Since \(Q\) is the reflection of \(P\) in \(HO\), we observe that \(P, T, Q\) are collinear, and \(PT = TQ\). Let \(QL, TN\) and \(OS\) be the perpendiculars drawn respectively from \(Q, T\) and \(O\) on to the altitude \(AK\). (See the figure.)

We have \(LN = NM\), since \(T\) is the mid-point of \(QP\); \(HN = NS\), since \(T\) is the mid-point of \(OH\); and \(HM = MK\), as \(P\) is the circum-centre of \(KHO\). We obtain

\[
LH + HN = LN = NM = NS + SM,
\]

which gives \(LH = SM\). We know that \(AH = 2OD\). Thus

\[
AL = AH - LH = 2OD - LH = 2SK - SM = SK + (SK - SM) = SK + MK
= SK + HM = SK + HS + SM = SK + HS + LH = SK + LS = LK.
\]

This shows that \(L\) is the mid-point of \(AK\) and hence lies on the line joining the midpoints of \(AB\) and \(AC\). We observe that the line joining the mid-points of \(AB\) and \(AC\) is also perpendicular to \(AK\). Since \(QL\) is perpendicular to \(AK\), we conclude that \(Q\) also lies on the line joining the mid-points of \(AB\) and \(AC\).
6. Define a sequence \( \langle a_n \rangle_{n \geq 0} \) by \( a_0 = 0, \ a_1 = 1 \) and
\[
a_n = 2a_{n-1} + a_{n-2},
\]
for \( n \geq 2 \).

(a) For every \( m > 0 \) and \( 0 \leq j \leq m \), prove that \( 2a_m \) divides \( a_{m+j} + (-1)^ja_{m-j} \).

(b) Suppose \( 2^k \) divides \( n \) for some natural numbers \( n \) and \( k \). Prove that \( 2^k \) divides \( a_n \).

Solution:

(a) Consider \( f(j) = a_{m+j} + (-1)^ja_{m-j}, \ 0 \leq j \leq m \), where \( m \) is a natural number. We observe that \( f(0) = 2a_m \) is divisible by \( 2a_m \). Similarly,
\[
f(1) = a_{m+1} - a_{m-1} = 2a_m
\]
is also divisible by \( 2a_m \). Assume that \( 2a_m \) divides \( f(j) \) for all \( 0 \leq j < l \), where \( l \leq m \). We prove that \( 2a_m \) divides \( f(l) \). Observe
\[
\begin{align*}
f(l-1) &= a_{m+l-1} + (-1)^{l-1}a_{m-l+1}, \\
f(l-2) &= a_{m+l-2} + (-1)^{l-2}a_{m-l+2}.
\end{align*}
\]
Thus we have
\[
a_{m+l} = 2a_{m+l-1} + a_{m+l-2} = 2f(l-1) - 2(-1)^{l-1}a_{m-l+1} + f(l-2) - (-1)^{l-2}a_{m-l+2} = 2f(l-1) + f(l-2) + (-1)^{l-1}(a_{m-l+2} - 2a_{m-l+1}) = 2f(l-1) + f(l-2) + (-1)^{l-1}a_{m-l}.
\]
This gives
\[
f(l) = 2f(l-1) + f(l-2).
\]
By induction hypothesis \( 2a_m \) divides \( f(l-1) \) and \( f(l-2) \). Hence \( 2a_m \) divides \( f(l) \). We conclude that \( 2a_m \) divides \( f(j) \) for \( 0 \leq j \leq m \).

(b) We see that \( f(m) = a_{2m} \). Hence \( 2a_m \) divides \( a_{2m} \) for all natural numbers \( m \). Let \( n = 2^k l \) for some \( l \geq 1 \). Taking \( m = 2^{k-1}l \), we see that \( 2a_m \) divides \( a_n \). Using an easy induction, we conclude that \( 2^ka_l \) divides \( a_n \). In particular \( 2^k \) divides \( a_n \).