Indian National Astronomy Olympiad – 2009

Senior Category — Roll Number: [ ]

Model Solutions

INAO – 2009 — Date: 31st January 2009

Duration: **Three Hours** — Maximum Marks: 100

Please Note:

- Please write your roll number on top of this page in the space provided.
- Before starting, please ensure that you have received a copy of the question paper containing total 3 pages (6 sides).
- In Section A, there are 10 multiple choice questions with 4 alternatives out of which only 1 is correct. You get 3 marks for each correct answer and **-1 mark for each wrong answer**.
- In Section B, there are 4 multiple choice questions with 4 alternatives each, out of which any number of alternatives may be correct. You get 5 marks for each correct answer. No marks are deducted for any wrong answers. **You will get credit for the question if and only if you mark all correct choices and no wrong choices.** There is no partial credit.
- For both these sections, you have to indicate the answers on the page 2 of the answersheet by putting a $\times$ in the appropriate box against the relevant question number, like this:

Q.NO.	(a)	(b)	(c)	(d)		Q.NO.	(a)	(b)	(c)	(d)
22	☐	☒	☐	☐	**OR**	35	☒	☒	☐	☐

  Marking a cross ($\times$) means affirmative response (selecting the particular choice). Do not use ticks or any other signs to mark the correct answers.
- In Section C, there are 5 analytical questions totalling 50 marks.
- Blank spaces are provided in the question paper for the rough work. No rough work should be done on the answer-sheet.
- No calculators are allowed.
- **The answer-sheet must be returned to the invigilator**. You can take this question booklet back with you.

HOMI BHABHA CENTRE FOR SCIENCE EDUCATION

Tata Institute of Fundamental Research

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Useful Physical Constants

Mass of the Earth	M_E	$\approx$	5.97×10^{24} kg
Radius of the Earth	R_E	$\approx$	6.4×10^{6} m
Mass of the Sun	$M_\odot$	$\approx$	1.99×10^{30} kg
Radius of the Sun	$R_\odot$	$\approx$	7×10^{8} m
Radius of the Moon	R_m	$\approx$	1.7×10^{6} m
Speed of Light	c	$\approx$	3×10^{8} m/s
Astronomical Unit	1 A. U.	$\approx$	1.5×10^{11} m
Gravitational Constant	G	$\approx$	6.67×10^{-11} m^3/(Kg s^2)
Gravitational Acceleration on the Earth	g	$\approx$	9.8 m/s^2
Gravitational Acceleration on the Moon	g_m	$\approx$	1.6 m/s^2

Space for Rough Work

Section A: (10 Q $\times$ 3 marks each)

1. If $a^x = b^y = c^z$ and $b^2 = ac$, then y = ?

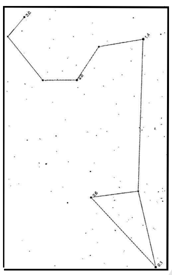
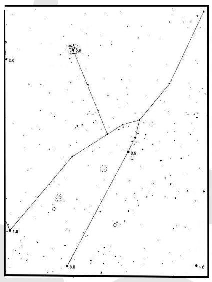
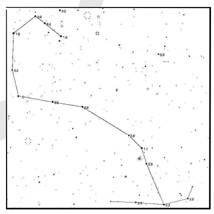
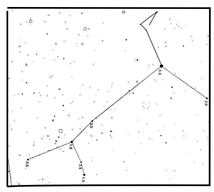
   (a) $\frac{2xz}{x+z}$ (b) $\frac{xz}{x+z}$ (c) $\sqrt{2xz}$ (d) $\sqrt{xz}$

Solution:

$$\begin{aligned} a &= b^{\frac{y}{x}} \\ c &= b^{\frac{y}{z}} \\ b^2 &= ac = b^{\frac{y}{x}} b^{\frac{y}{z}} \\ &= b^{\frac{y}{x}+\frac{y}{z}} \\ 2 &= \frac{y}{x} + \frac{y}{z} \\ &= \frac{y(x+z)}{xz} \\ y &= \frac{2xz}{z+x} \end{aligned}$$

Ans = **(a)**

2. Each of the figures below depict a constellation. Find the odd one out.

(a)  (b)  (c)  (d) 

Solution: The constellations are (a) Leo (b) Taurus (c) Scorpio (d) Canis Major. First three are zodiac signs whereas the fourth one is not.
Ans = **(d)**

3. Gravitational force between two identical uniform solid gold spheres of radius r each in contact is proportional to

   (a) r^4 (b) r^2 (c) $\frac{1}{r^2}$ (d) $\frac{1}{4r^2}$

Solution: The distance between two spheres $= 2r$ and masses are the same

$$\begin{aligned}
\Rightarrow F &= \frac{GM^2}{(2r)^2} \\
&= \frac{G(\frac{4\pi r^3}{3} * \rho)^2}{4r^2} \\
\Rightarrow F &\propto \frac{r^6}{r^2} \\
F &\propto r^4
\end{aligned}$$

Thus ans = **(a)**

4. A copper cube and a wooden cube of volume 10^{-3} m^3 each are initially at room temperature. They are then moved to an enclosure of ambient temperature 50°C. What can we conclude about the temperatures attained by both cubes after 5 hours?
   (a) $T_{\text{copper}} > T_{\text{wood}}$ as thermal conductivity of copper is greater than that of wood.
   (b) $T_{\text{wood}} > T_{\text{copper}}$ as specific heat capacity of wood is greater than that of copper.
   (c) The temperatures will depend on the interplay between specific heat capacity and thermal conductivity of the materials.
   (d) Both temperatures will be practically the same, as they are in the enclosure for 5 hours.

Solution: Both copper and wooden cube will have same temperature, as that of the enclosure, because the time is sufficiently long to bring them in thermal equilibrium with their surroundings.
Ans = **(d)**

5. If the product of all the numbers from 1 to 100 is divisible by 2^n, then what is the maximum possible value for n?

   (a) 128 (b) 97 (c) 64 (d) 87

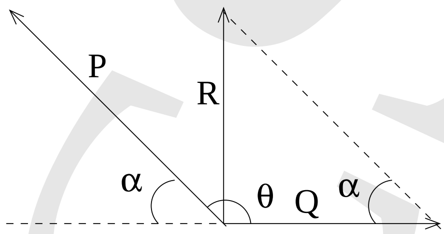
Solution: There are 50 numbers between 1 to 100, which are divisible by at least first power of 2.
There are 25 numbers, which are divisible by at least second power of 2, i.e. 4. However, as they are already counted once in previous step, we count them again for only single power of 2 in this step.
Continuing in the same fashion, there are 12 numbers, which are divisible by at least third power of 2, i.e. 8.

There are 6 numbers, which are divisible by at least forth power of 2, i.e. 16.
There are 3 numbers, which are divisible by at least fifth power of 2, i.e. 32.
There is only 1 number divisible by sixth power of 2, i.e. 64.
Summing,
$(50 + 25 + 12 + 6 + 3 + 1) = 97$ is the number of times factor 2 appears in the product.
So 2^{97} is the highest power of 2, which will be a factor of the product of all the numbers from 1to 100.
Ans = **(b)**

6. Two vectors $\overrightarrow{P}$ and $\overrightarrow{Q}$ are acting at a point such that their resultant is perpendicular to $\overrightarrow{Q}$. If θ is the angle between $\overrightarrow{P}$ and $\overrightarrow{Q}$ then $\frac{|P|}{|Q|}$ is given by,

   (a) $\cos\theta$ (b) $\sec\theta$ (c) $-\cos\theta$ (d) $-\sec\theta$

Solution: See figure



$$
\begin{aligned}
\alpha &= 180 - \theta \\
\frac{Q}{P} &= \cos\alpha \\
&= \cos(180 - \theta) \\
&= -\cos\theta \\
\therefore \frac{P}{Q} &= -\sec\theta
\end{aligned}
$$

7. What will be the approximate period of Chandrayaan moving in an orbit 100 km above the moon's surface?

   (a) 57 min (b) 30 min (c) 118 min (d) 79 min

Solution: Let R = radius of the Moon, r = radius of the orbit
Using Kepler's Law,

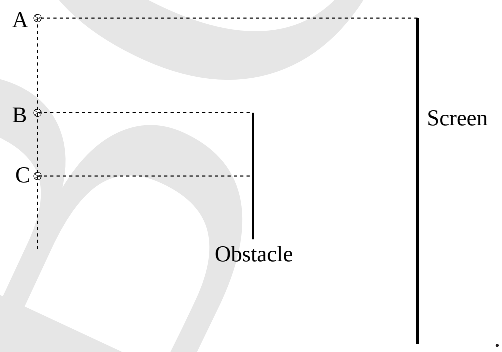
$$
\begin{aligned}
T^2 &= \frac{4\pi^2}{GM_m} r^3 \\
T^2 &= \frac{4\pi^2}{gR^2} r^3
\end{aligned}
$$

Radius of orbit of Chandrayaan from center of the moon is, $r = 1.7{\times}10^6 m{+}100km$

$$\begin{aligned}
T^2 &= \frac{4\pi^2}{1.6 \times (1.7 \times 10^6)^2}(1.8 \times 10^6)^3 \\
&\approx \frac{4 \times 10 \times 1.8^3}{1.6 \times 1.7^2} \times 10^6 \\
&\approx \frac{45 \times 1.8^2}{1.7^2} \times 10^6 \\
T &\approx \frac{6.7 \times 18}{17} \times 10^3 \\
&\approx \left(6.7 + \frac{6.7}{17}\right) \times 10^3 \\
&\approx 7100\text{sec} \\
T &\approx 118\text{min.}
\end{aligned}$$

is the period of Chandrayaan orbit. The calculation above is a back of the envelope calculation to estimate the period.
Ans = **(c)**

8. In the following figure, A, B, C are three light source positions with respect to the obstacle and the screen. Which of these light source positions will result in the longest shadow of the obstacle on the screen?



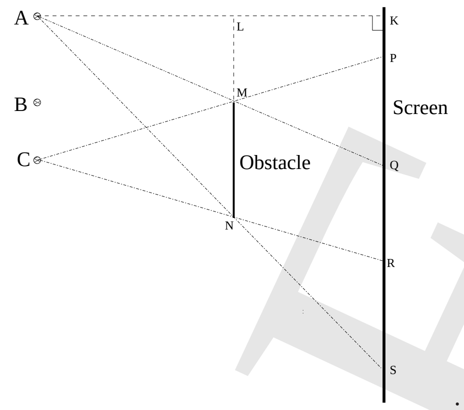
(a) A

(b) C

(c) A and C form shadows of same length, while B forms a smaller shadow.

(d) All the three light sources will form shadows of same length.

Solution: Ans = **(d)**



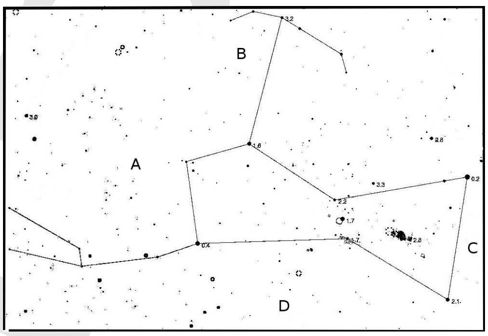
From the above figure, PR is the shadow because of light source C and QS is the shadow because of the light source A.

Now line AK is perpendicular to screen from A, which meets line MN at L.
Since $MN \parallel KS$,
$\therefore \triangle\ AMN\ and\ \triangle\ AQS$ are congruent ,
which means $\dfrac{MN}{QS} = \dfrac{AL}{AK}$.

Similarly, $\triangle CMN$ & $\triangle\ CPR$ are congruent,
which means $\dfrac{MN}{PR} = \dfrac{AL}{AK}$.
$\Longrightarrow$ QS=PR.

Therefore, shadows of the obstacle formed by either light source A or C are the same. In the same manner, we can prove that the shadow formed by the light source B is also of the same length.

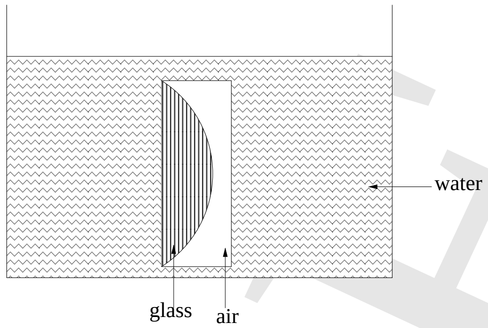
9. The following figure shows skeleton chart of the Orion constellation. Approximate direction "North" is marked with the letter ....



(a) A (b) B (c) C (d) D

Solution: The Orion's head is northwards.
Ans = **(a)**

10. Find the resultant focal length for following system where the radius of curvature is 15cm.



(a) 40 cm (b) 60 cm (c) 120 cm (d) ∞

Solution: Refractive Index : $n_w = \frac{4}{3}$; $n_g = \frac{3}{2}$

Refractive Index of glass relative to water: ${}_w n_g = \dfrac{n_g}{n_w} = \dfrac{9}{8}$

Refractive Index of air relative to water: ${}_w n_a = \dfrac{n_a}{n_w} = \dfrac{3}{4}$

Using the lens maker formula,

$$\frac{1}{f} = (n-1)(\frac{1}{R_1} + \frac{1}{R_2})$$

focal length for plano-convex glass-in-water lens, where $R_1 = \infty; R_2 = 15cm$ is,

$$\frac{1}{f_1} = ({}_w n_g - 1)(\frac{1}{R_1} + \frac{1}{R_2}) = \frac{(\frac{9}{8} - 1)}{15} = \frac{1}{120}$$

$\therefore f_1 = 120$ cm

Similarly, focal length for plano-concave air-in-water lens, where R_1 = -15cm; $R_2 = \infty$ is,

$$\frac{1}{f_2} = ({}_w n_a - 1)(\frac{1}{R_1} + \frac{1}{R_2}) = \frac{(\frac{3}{4} - 1)}{-15} = \frac{1}{60}$$

$\therefore f_2 = 60$ cm

Hence resultant focal length will be

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} = (\frac{1}{120} + \frac{1}{60}) = \frac{1}{40}$$

$\therefore f = 40$ cm. Ans is **(a)**.

Section B: (4 Q × 5 marks each)

11. Which of the following observations support the statement that "Every system tends to configure itself to have minimum Potential Energy".
    (a) Andromeda galaxy and Milky Way are approaching each other.
    (b) Two unlike, free charges move towards each other.
    (c) External work is required to compress a spring.
    (d) Light iron dust moves towards powerful magnet in close vicinity.

Solution: Ans = **(b)**, **(c)** and **(d)**.
Andromeda Galaxy and the Milky Way are moving around the centre of our local group of galaxies. In the course of this motion, they just happen to be coming closer to each other. Their mutual gravitational attraction does not play any significant role in this motion.

12. Consider a sealed frictionless piston cylinder assembly where the piston mass and atmospheric pressure above the piston remain constant. A gas in the cylinder is heated and hence it expands. Which of the following is / are true?
    (a) The density of the gas will increase.
    (b) The pressure of the gas will decrease.
    (c) The internal energy of the system will remain the same.
    (d) In this process work is done by the gas.

Solution: As gas in the cylinder is heated and it expands so work is done by the gas. At the new equilibrium, the density of the gas would have decreased (same mass in larger volume), internal energy would have increased (proportional to change in temperature) and the pressure would have remained the same (as that of atmospheric pressure + pressure due to piston mass).
Ans = **(d)**.

13. In one of the truly revolutionary finds of the 20^{th} century, Howard Carter discovered the tomb of the Egyptian Pharaoh (emperor) Tutankhamun in 1922. Following items were removed from the tomb, along with the mummy of the Pharaoh. Which of these items could have been carbon dated to fix the period of the Pharaoh?
    (a) Fragments of glass
    (b) Golden Bracelets
    (c) Dried Fruits
    (d) Leather Shoe

Solution: Carbon dating relies on the fact that all living objects have ^{12}C and ^{14}C in a fixed ratio to each other. The radioactive ^{14}C keeps decaying to ^{12}C, however, it is replaced by food intake consumed by living organisms. After death, the decay process continues, but there is no replacement for decayed ^{14}C. Thus the ratio starts changing.
In the list above, the first two are not made from any organic / living substances. Thus, they cannot be used for carbon dating.
Ans = **(c)** and **(d)**.

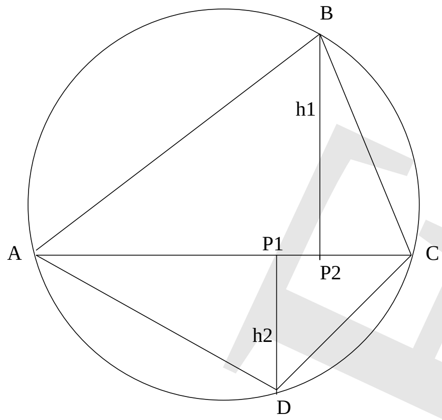
14. Which of the following phenomena is / are useful, in estimating distances in the Universe?
    (a) Some time Venus can be seen transiting over the solar disc.
    (b) Stars with no proper motion appear to change their position in the sky when viewed six months apart.
    (c) Stars exhibit Doppler shift.
    (d) All supernovae of Type Ia have same absolute brightness.

Solution: With Doppler shift we can estimate the velocity of stars but not the distance.
The Earth-Sun distance was successfully estimated for the first time using Venus transit method. Option (b) talks of parallax method. The absolute magnitudes of Supernovae is useful standard candle for cosmological distances.
Hence, ans= **(a), (b)** and **(d)**.

Section C: Analytical Questions

α. (8 marks) What will be area of the largest cyclic quadrilateral that can be inscribed in a given circle? Justify your answer qualitatively (formal proof not necessary).

Solution: Divide any quadrilateral ABCD, inscribed in a circle, into two triangles $\triangle ABC$ and $\triangle ADC$, as shown.



The area of the triangle is given by the formula,

$$\frac{1}{2} \times base \times height.$$

The common base of the two triangles $\triangle ABC$ and $\triangle ADC$ is given by $\overline{AC}$.
The total area, of the quadrilateral will be sum of the areas of the two triangles i.e.,

$$\frac{1}{2} \times l(\overline{AC}) \times (h_1 + h_2)$$

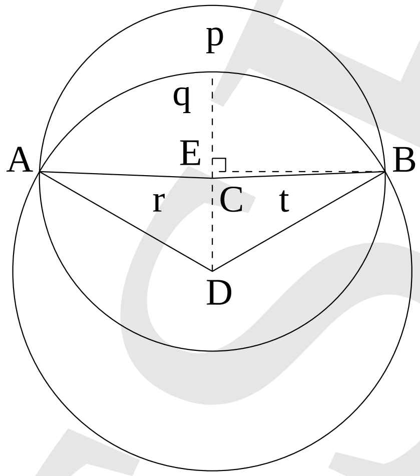
To maximize the total area, $l(\overline{AC})$ as well as $(h_1 + h_2)$ should be maximized.
Now the maximum possible length that can be fitted inside a circle has to be its diameter, d.
Consequently, the total area of the quadrilateral ABCD would be maximum if, $(h_1 + h_2) = d$ and $l(\overline{AC}) = d$
Thus, the maximum area can be $\frac{1}{2}d^2$.
Noting that the the base and height are perpendicular to each other, it is clear that the said quadrilateral is a square.

- If correct area with no justification: **2 marks**.
- If correct area with incorrect justification: **3 marks**.
- If justification only considers cyclic rectangles: **4 marks**.
- If correct justification, but minor error resulting in wrong area: **7 marks**.
- Any correct method will receive full consideration.

β. (12 marks) Jayshree claimed that she saw a solar eclipse when the size of the solar disk was $26'$ and that of the lunar disk was $30'$. She also claimed that at the time of the maximum eclipse, distance between the centres of the two disks was $7'$. Qualitatively show that she could not have observed a total eclipse. Find the percentage of the solar disk covered at the time of the maximum eclipse. (Given: $\cos^{-1}\left(\frac{1}{26}\right) \approx 0.49\pi$

rad).

Solution: At the time of the maximum eclipse, the centres were $7'$ away from each other. However, the radius of the solar disk is smaller than that of the lunar disk by just $2'$. Thus, she must not have viewed the total solar eclipse. Let us find percentage of the maximum partial eclipse. **(1 mark)**



(1 mark)

In the figure above, points A and B make an angle of θ_1 and θ_2 with centre of the Solar and the Lunar disks (i.e. C and D) respectively.

r_s denotes the solar radius, r_m the lunar radius and h the separation between the two centres at the closest approach (all in arcminutes).

$$\begin{aligned} A(\widehat{ACB}) &= \pi r_s^2 \frac{\theta_1}{2\pi} \\ p + q &= \frac{r_s^2 \theta_1}{2}. \end{aligned}$$

$$\begin{aligned} A(\widehat{ADB}) &= \pi r_m^2 \frac{\theta_2}{2\pi} \\ q + r + t &= \frac{r_m^2 \theta_2}{2}. \end{aligned}$$

where r = $A(\triangle ACD)$ and t = $A(\triangle BCD)$
Also from figure, $r = t$.

$$\Rightarrow p = \frac{r_s^2 \theta_1}{2} - \frac{r_m^2 \theta_2}{2} + 2A(\triangle ACD). \qquad \text{(2 marks)}$$

Now using Hero's formula for area of a triangle,

$$
\begin{aligned}
A(\triangle ACD) &= \sqrt{s(s-r_m)(s-r_s)(s-h)} \\
\text{where,} & \\
r_s &= 13' \\
r_m &= 15' \\
h &= 7' \\
s &= \frac{r_s + r_m + h}{2} \\
\therefore s &= \frac{13+15+7}{2} = \left(\frac{35}{2}\right)' \\
\therefore \mathrm{A}(\triangle \mathrm{ACD}) &= \sqrt{\frac{35}{2} \times \frac{(35-30)}{2} \times \frac{(35-26)}{2} \times \frac{(35-14)}{2}} \\
&= \sqrt{\frac{35}{2} \times \frac{5}{2} \times \frac{9}{2} \times \frac{21}{2}} \\
&= \sqrt{\left(\frac{5 \times 7 \times 3}{4}\right)^2 \times 3} \\
&= \frac{105}{4} \times \sqrt{3} \\
&\approx \frac{(173.2 + 8.7)}{4} \\
&\approx \frac{91}{2} \qquad \text{(2 marks)}
\end{aligned}
$$

Now to find $\theta_1$and θ_2 draw a perpendicular line BE to line CD.
Let EC = x and EB = y.

$$
\begin{aligned}
\therefore In \triangle ECB, & \\
x^2 + y^2 &= r_s^2. \\
\text{and in } \triangle EDB, & \\
(x+h)^2 + y^2 &= r_m^2 \\
2hx + h^2 + r_s^2 &= r_m^2 \\
\therefore x &= \frac{r_m^2 - r_s^2 - h^2}{2h} = 0.5' \\
\therefore cos\theta_1 &= \frac{x}{r_s} = \frac{0.5}{13} = \frac{1}{26} \\
\Rightarrow \theta_1 &= 0.49\pi \qquad \text{(2 marks)} \\
\text{Similarlly,} & \\
cos\theta_2 &= \frac{x+h}{r_m} = \frac{0.5+7}{15} = \frac{1}{2} \\
\Rightarrow \theta_2 &= \frac{\pi}{3} \qquad \text{(1 mark)}
\end{aligned}
$$

These are half angles only. Total angles are double these values.

$$\begin{aligned}
\Rightarrow p &= \frac{r_s^2 \times 2 \times \theta_1}{2} - \frac{r_m^2 \times 2 \times \theta_2}{2} + 2A(\triangle ACD) \\
&\approx 169 \times 0.49\pi - 225 \times \frac{\pi}{3} + 2 \times \frac{91}{2} \\
&\approx \pi\left(84 - 75 + \frac{91 \times 7}{22}\right) \\
&\approx \pi\left(9 + \frac{637}{22}\right) \\
&\approx \pi(9 + 28.95) \\
&\approx 38\pi \text{ arcmin}^2 \qquad \text{(1.5 marks)}
\end{aligned}$$

$$\begin{aligned}
\text{and } A(Sun) &= \pi r_s^2 \\
&= 169\pi \text{ arcmin}^2 \qquad \text{(0.5 marks)}
\end{aligned}$$

$\therefore$ The amount of solar surface covered by moon at the time of maximum eclipse

$$\begin{aligned}
&= (1 - \frac{p}{A(Sun)}) \times 100\% \\
&\approx (1 - \frac{38}{169}) \times 100\% \\
&\approx (1 - \frac{2.923}{13}) \times 100\% \\
&\approx (1 - 0.22) \times 100\%. \\
&\approx 78\ \%. \qquad \text{(1 mark)}
\end{aligned}$$

γ. (8 marks) The famous Indian astronomer, Aryabhata, expressed the value of π in what we now know as continuing fractions i.e..$\pi = 3.1416 = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d}}}$ where a, b, c, d are positive integers. Find a, b, c, d.

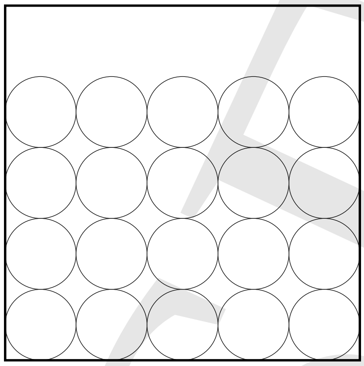
Solution:

$$
\begin{aligned}
3.1416 &= 3 + \frac{1416}{10000} \\
&= 3 + \cfrac{1}{\cfrac{10000}{1416}} \\
&= 3 + \cfrac{1}{\cfrac{9912+88}{1416}} \\
&= 3 + \cfrac{1}{7 + \cfrac{1}{\cfrac{1416}{88}}} \\
&= 3 + \cfrac{1}{7 + \cfrac{1}{\cfrac{1408+8}{88}}} \\
&= 3 + \cfrac{1}{7 + \cfrac{1}{16 + \cfrac{1}{\cfrac{88}{8}}}} \\
&= 3 + \cfrac{1}{7 + \cfrac{1}{16 + \cfrac{1}{11}}}
\end{aligned}
$$

Therefore $a = 3$, $b = 7$, $c = 16$, $d = 11$.

- If attempted to solve polynomial: **1 mark**.
- If approximate reciprocals found correctly: **6 marks**.
- If approximate reciprocals found correctly and final answer is tallied back: **full marks**.
- For each wrong value out of a, b, c, d: **−1 mark each**.

δ. (10 marks) Kedar sent a container of marbles by road from Mumbai to Parag in Pune. The container was $2m \times 2m \times 2.5m$ in size, with height being the larger dimension. Marbles of 2 cm diameter were arranged to fill the entire base and then additional layers of marbles were arranged with each upper marble exactly on top of corresponding marble in the previous layer (see figure). The marbles were thus placed upto 2m height to complete the cubical structure. However, on reaching Pune, when Parag opened the container, he found the height of structure was not 2m as promised by Kedar, but something else. Kedar defended saying that marbles may have readjusted due to jiggling. Can you find new height of this marble pile?



Solution: Total number of marbles is $100 \times 100 \times 100 = 10^6$ arranged in total 100 layers. The marbles will rearrange themselves so as to occupy vacant spots in the structure. We assume that there is enough jiggling in the trip from Mumbai to Pune so that uniformity is attained in the arrangement of the marbles throughout the box.

There are three ways to solve this problem and all of them lead to nearly the same answer. (**All three methods earn full credit**).

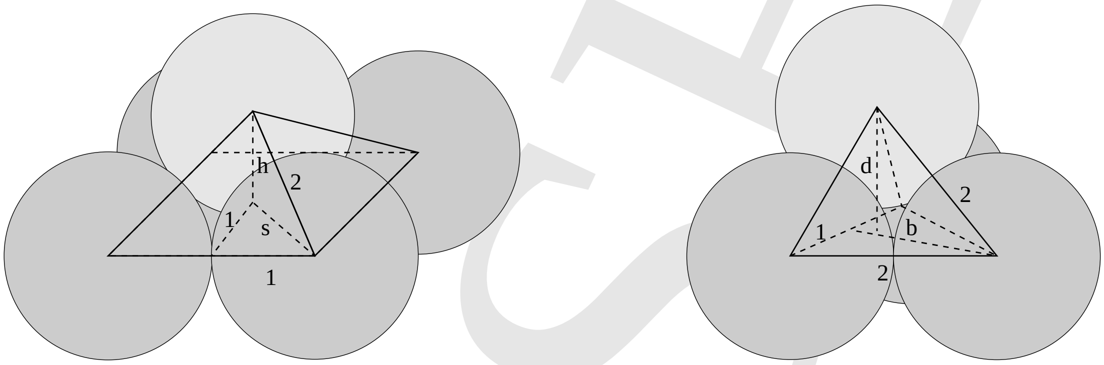
Method 1: The marbles will be rearranged in a "face-centred cubic" structure. In this structure, the bottom layer has 100×100 marbles, arranged such that each 4 marbles create a gap between them, into which a marble can be placed in the next layer. Since there are 99×99 such gaps, the next layer will consist of 99×99 marbles. The layer above that will again have 100 marbles, and so on. Thus, we will have 50 layers of 100×100 marbles and 50 layers of 99×99 marbles. But in addition, we shall have extra marbles displaced from the 99×99 layers. Number of marbles displaced in each such layer is $(100^2 - 99^2) = 199$. There are total 50 such layers. Thus, total number of marbles remaining after filling 100 layers is

$$
\begin{aligned}
n &= 199 \times 50 \\
&= 200 \times 50 - 50 \\
&= (100 \times 100) - 50
\end{aligned}
$$

The 101^{st} layer can be of 100×100 marbles. The number of marbles remaining are in fact smaller than this number. Thus, there will be one additional incomplete layer.

In this formation each set of 4 marbles on one layer and the 1 marble on the next layer sitting in the gap between them form a pyramid. Let the height of this pyramid calculated from the plane containing the centres of the 4 marbles in the first layer to the centre of the top marble be h cm. Since the radius of each marble is 1 cm, according to the left panel in the figure below,

$$\begin{aligned} h &= \sqrt{2^2 - s^2} \\ &= \sqrt{4 - (1^2 + 1^2)} \\ &= \sqrt{2} \end{aligned}$$



Now,
Distance between floor and centre of marbles in the 1$^{\text{st}}$ layer is 1 cm,
Vertical separation between 1$^{\text{st}}$ and 2$^{\text{nd}}$ layer centres is $\sqrt{2}$ cm,
Vertical separation between 2$^{\text{nd}}$ and 3$^{\text{rd}}$ layer centres is $\sqrt{2}$ cm,
and so on.
For 101$^{\text{st}}$ layer, distance between the top and marble centres will again be 1 cm.

Thus, total height of the structure will be,

$$\begin{aligned} H &= 1 + 100 \times \sqrt{2} + 1 \text{ cm} \\ &\approx 1 + 141.4 + 1 \text{ cm} \\ &\approx 143.4 \text{ cm} \end{aligned}$$

Method 2: The marbles will be rearranged in a "hexagonal close-packed" structure. In this structure, the bottom layer has 100 marbles in the first row next to a wall, but the adjacent row has 99 marbles, fitting into the gaps of the first row. The next row will have again 100 marbles, and so on. The distance between the lines connecting the centres of one row of marbles to the next row is (see right panel of figure above)

$$\sqrt{2^2 - 1^2} \text{ cm} = \sqrt{3} \text{ cm}.$$

The distance from the centre of the first row to the wall is 1 cm, and the distance from the centre of the last row to the opposite wall is 1 cm. Therefore if the number of rows between opposite walls is r, then

$$1 + (r - 1)\sqrt{3} + 1 = 200$$

Solving, we get $r = 198/1.73 + 1 \approx 115$ (r must be rounded off to the nearest *lower* integer, since the walls are fixed). Thus one can put 58 rows of 100 marbles and 57 rows of 99 marbles in the first layer. That is a total of $58 \times 100 + 57 \times 99 = 100 + 57 \times 100 \times 2 - 57 = 11443$ marbles in the first layer.

The second layer can be arranged in exactly the same manner, but now each row will fit in the gaps created by the adjacent rows in the first layer. Since the pattern of gaps is reverse of the pattern of marbles in the first layer, one can put 58 rows of 99 marbles and 57 rows of 100 marbles in the second layer. That is a total of $58 \times 99 + 57 \times 100 = 58 \times 100 \times 2 - 58 - 100 = 11442$ marbles in the second layer.

Thus the first two layers will consist of $11443 + 11442 = 22885$ marbles. From the third layer, the pattern will repeat. Therefore, we shall have $10^6/22885 \approx 44$ (rounding off to the nearest *larger* integer) pairs of layers, i.e., 88 layers in all (the last layer being incomplete).

In this formation each set of 3 marbles on one layer and the 1 marble on the next layer sitting in the gap between them form a tetrahedron. Let the height of this tetrahedron calculated from the plane containing the centres of the 3 marbles in the first layer to the centre of the top marble be d cm. Since the radius of each marble is 1 cm, according to the right panel in the figure below,

$$\begin{aligned} d &= \sqrt{2^2 - b^2} \\ &= \sqrt{4 - (\frac{2}{3}\sqrt{2^2 - 1^2})^2} \\ &= \sqrt{4 - (\frac{2}{3}\sqrt{3})^2} \\ &= \sqrt{4 - \frac{4}{3}} \\ &= 2\sqrt{2}/\sqrt{3} \end{aligned}$$

Now,
Distance between floor and centre of marbles in the 1^{st} layer is 1 cm,
Vertical separation between 1^{st} and 2^{nd} layer centres is $2\sqrt{2}/\sqrt{3}$ cm,
Vertical separation between 2^{nd} and 3^{rd} layer centres is $2\sqrt{2}/\sqrt{3}$ cm,
and so on.
For 88^{th} layer, distance between the top and marble centres will again be 1 cm.

Thus, total height of the structure will be

$$\begin{aligned} H &= 1 + \frac{87 \times 2\sqrt{2}}{\sqrt{3}} + 1 \text{ cm} \\ &\approx 1 + \frac{174 \times 1.414}{1.732} + 1 \text{ cm} \\ &\approx 2 + 100 \times 1.414 \text{ cm} \\ &\approx 144 \text{ cm} \end{aligned}$$

Method 3: The labour involved in the above two solutions can be avoided to a great extent by recognising that the two arrangements described above are, in fact, the "face-centred cubic" and "hexagonal close packing" arrangements. In each of these cases, the "packing fraction" is 0.74, i.e., the actual volume occupied by the marbles is 74% of the apparent volume of the stack. Therefore, if V is the volume occupied in each of these arrangements, then V is given by the equation

$$0.74V = 10^6 \times \frac{4\pi}{3}(1\text{cm})^3 \tag{1}$$

Now, $V = A \times H$, where A is the area of the stack. Here, $A = 200\,\text{cm}^2$. Thus,

$$\begin{aligned} H &= V/A \\ &= \frac{10^6 \times \frac{4\pi}{3}}{0.74 \times 4 \times 10^4}\ \text{cm} \\ &\approx \frac{100 \times 3.1416}{3 \times 0.74}\ \text{cm} \\ &\approx \frac{100 \times 104.72}{74}\ \text{cm} \\ &\approx 141.4\ \text{cm} \end{aligned}$$

We obtain an answer close to those of the first two methods. The small discrepancy among the three methods arises due to the difference in packing of the balls near the edges, i.e., close to the walls (which manifests itself in the approximations that we have to make in each case). The answer obtained in the third method is the ideal case scenario, where we did not consider any "edge effect".

(In the last method, recognising that 0.74 is actually $\frac{\pi}{3\sqrt{2}}$ will simplify the calculation.)

ϵ. Chiraag performed an experiment using a simple pendulum to find value of g. He measured time taken for 30 oscillations of the pendulum for various values of length (repeated thrice for each value of length), following readings were obtained.

L	Readings		
cm	t_1 (Sec)	t_2 (Sec)	t_3 (Sec)
20.0	26.9	26.9	27.0
25.0	30.1	30.1	30.1
30.0	32.9	32.8	32.7
35.0	35.6	35.8	35.7
40.0	38.0	38.1	38.1
45.0	40.4	40.4	40.5

Mass of the bob of the pendulum was known to be 50 gm.

(a) (9 marks) Plot appropriate graph to represent the data.

(b) (2 marks) Find the value of g.

(c) (1 mark) Using the graph, find out required length of the pendulum to get a time period of 1.24 sec.

Solution: The equation for period of simple pendulum is given as,

$$T = 2\pi\sqrt{\frac{L}{g}}$$

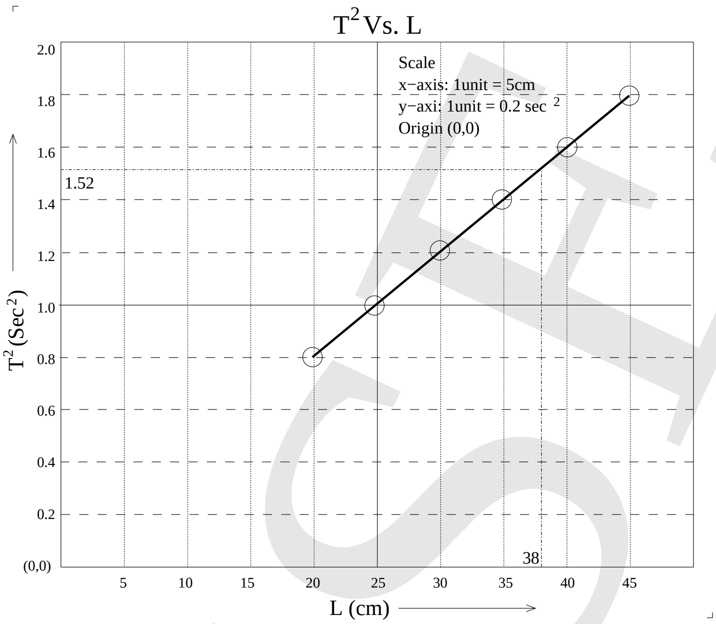
$$\therefore T^2 = \frac{4\pi^2}{g}L$$

In the experiment above, time is measured for 30 oscillation, So we have to average out three readings for each length and then find time for single oscillation.

L	Readings			t_{avg}	T=$\frac{t}{30}$	T^2
(cm)	t_1 (Sec)	t_2 (Sec)	t_3 (Sec)	(Sec)	(Sec)	Sec^2
20.0	26.9	26.9	27.0	26.9	0.897	0.81
25.0	30.1	30.1	30.1	30.1	1.03	1.06
30.0	32.9	32.8	32.7	32.8	1.09	1.19
35.0	35.6	35.8	35.7	35.6	1.19	1.41
40.0	38.0	38.1	38.1	38.1	1.27	1.61
45.0	40.4	40.4	40.5	40.5	1.35	1.82

- Data should be properly averaged converted to time period for one oscillation. **(1.5 marks)**
- Proper choice of variable on x-axis. **(1 mark)**
- Optimum amount of graph paper should be utilized.
- T^2 values should be computed and linear graph should be plotted. (Graph of T vs. $\sqrt{L}$ is also acceptable). **(2 marks)**
- Choice of scale should be convenient. **(0.5 mark)**
- Points should be clearly marked. **(1 mark)**
- Chart Title, Axis Titles etc. should be properly written. **(0.5 marks)**
- The scales and preferably origin should be clearly specified at the top right corner. **(0.5 marks)**
- Points used for finding slope should be well separated, well marked and preferably not from the existing dataset. **(0.5 marks)**
- Line should visually appear to be well balanced. **(1 mark)**
- 38cm point should be marked on the graph paper (as shown). **(0.5 marks)**

The T^2 Vs. L graph should be plotted to find slope.



From the graph,

$$\begin{aligned}
\text{slope} &\approx \frac{(1.61 - .79)}{(40 - 20)} \\
&\approx \frac{(.82)}{(20)} = 4.1 \times 10^{-2} \\
\therefore \frac{4\pi^2}{g} &\approx 4.1 \times 10^{-2} \\
\therefore g &\approx \frac{40}{4.1 \times 10^{-2}} \\
g &\approx 9.8 \text{ m/s}^2 \qquad \text{(2 mark)}
\end{aligned}$$

Using the graph, when $T = 1.24$ sec, $T^2 = 1.54$ sec^2. Thus, $L \approx 38$ cm. **(1 mark)**